PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

# On the Mechanical Conditions of a Swarm of Meteorites, and on Theories of Cosmogony

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# PHILOSOPHICAL TRANSACTIONS.

## I. *On the Mechanical Conditions of a Swarm of Meteorites, and on Theories of Cosmogony.*

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Mr. Lockyer writes in his interesting paper on Meteorites* as follows :—

"The brighter lines in spiral nebulæ, and in those in which a rotation has been set up, are in all probability due to streams of meteorites with irregular motions out of the main streams, in which the collisions would be almost *nil*. It has already been suggested by Professor G. Darwin ('Nature,' vol. 31, 1884–5, p. 25)—using the gaseous hypothesis—that in such nebulæ 'the great mass of the gas is non-luminous, the luminosity being an evidence of condensation along lines of low velocity, according to a well known hydrodynamical law. From this point of view, the visible nebula may be regarded as a luminous diagram of its own stream-lines.'"

The whole of Mr. Lockyer's paper, and especially this passage in it, leads me to make a suggestion for the reconciliation of two apparently divergent theories of the origin of planetary systems.

The nebular hypothesis depends essentially on the idea that the primitive nebula is a rotating mass of fluid, which at successive epochs becomes unstable from excess of rotation, and sheds a ring from the equatorial region.

The researches of Roche† (apparently but little known in this country) have imparted to this theory a precision which was wanting in Laplace's original exposition, and have rendered the explanation of the origin of the planets more perfect.

But notwithstanding the high probability that some theory of the kind is true,‡ the acceptance of the nebular hypothesis presents great difficulties.

Sir William Thomson long ago expressed to me his opinion that the most probable origin of the planets was through a gradual accretion of meteoric matter, and

* 'Nature,' Nov. 17, 1887. The paper itself is in 'Roy. Soc. Proc.,' Nov. 15, 1887 (No. 259, p. 117).

† 'Montpellier, Acad. Sci. Mém.'

‡ [The very remarkable photograph of the nebula in Andromeda, exhibited to the Royal Astronomical Society by Mr. Isaac Roberts on December 6, 1888, affords something like a proof of the substantial truth of the nebular hypothesis.—G. H. D. December 19, 1888.]

the researches of Mr. LOCKYER afford actual evidence in favour of the abundance of meteorites in space.

But the very essence of the nebular hypothesis is the conception of fluid pressure, since without it the idea of a figure of equilibrium becomes inapplicable. Now, at first sight, the meteoric condition of matter seems absolutely inconsistent with a fluid pressure exercised by one part of the system on another. We thus seem driven either to the absolute rejection of the nebular hypothesis, or to deny that the meteoric condition was the immediate antecedent of the Sun and Planets. M. FAYE has taken the former course, and accepts as a necessary consequence the formulation of a succession of events quite different from that of the nebular hypothesis.* I cannot myself find that his theory is an improvement on that of LAPLACE, except in regard to the adoption of meteorites, for he has lost the conception of the figure of equilibrium of a rotating mass of fluid.

The object of this paper is to point out that by a certain interpretation of the meteoric theory we may obtain a reconciliation of these two orders of ideas, and may hold that the origin of stellar and planetary systems is meteoric, whilst retaining the conception of fluid pressure.

According to the kinetic theory of gases, fluid pressure is the average result of the impacts of molecules. If we imagine the molecules magnified until of the size of meteorites, their impacts will still, on a coarser scale, give a quasi-fluid pressure. I suggest then that the fluid pressure essential to the nebular hypothesis is, in fact, the resultant of countless impacts of meteorites.

The problems of hydrodynamics could hardly be attacked with success, if we were forced to start from the beginning and to consider the cannonade of molecules. But when once satisfied that the kinetic theory will give us a gas, which, in a space containing some millions of molecules, obeys all the laws of an ideal non-molecular gas filling all space, we may put the molecules out of sight and treat the gas as a plenum.

In the same way, the difficulty of tracing the impacts of meteorites in detail is insuperable; but, if we can find that such impacts give rise to a quasi-fluid pressure on a large scale, we may be able to trace out many results by treating an ideal plenum. LAPLACE'S hypothesis implies such a plenum, and it is here maintained that this plenum is merely the idealisation of the impacts of meteorites.

As a bare suggestion this view is worth but little, for its acceptance or rejection must turn entirely on numerical values, which can only be obtained by the consideration of some actual system. It is obvious that the solar system is the only one about which we have sufficient knowledge to afford a basis for discussion. This paper is accordingly devoted to a consideration of the mechanics of a swarm of meteorites, with special numerical application to the solar system.

The investigation has entailed a considerable amount of mathematical analysis;

* 'Sur l'Origine du Monde,' Paris, GAUTHIER-VILLARS, 1884; 'Annuaire pour l'an 1885, Bureau des Longitudes,' p. 757.

there is, however, no analysis in §§ 1 and 2. The reader who only wishes to know the arguments and results, without a consideration of the mathematical details, is therefore recommended, after reading §§ 1 and 2, to pass on to the Summary.

## § 1. *On the Effective Elasticity of Meteorites in Collision.*

When two meteoric stones meet with planetary velocity, the stress between them during impact must generally be such that the limits of true elasticity are exceeded; and it may be urged that a kinetic theory is inapplicable unless the colliding particles are highly elastic. It may, however, I think, be shown that the very greatness of the velocities will impart what virtually amounts to an elasticity of a high order of perfection.

It appears, *a priori*, probable that, when two meteorites clash, a portion of the solid matter of each is volatilised, and Mr. LOCKYER considers the spectroscopic evidence conclusive that it is so. There is, no doubt, enough energy liberated on impact to volatilise the whole of both bodies, but only a small portion of each stone will undergo this change.

A rough numerical example will show the kind of quantities with which we are here dealing.

It will appear hereafter that the mean velocity of a meteorite may be at the least about 5 kilometres a second; and, accordingly, the mean relative velocity of a pair would then be about 7 kilometres a second.* Hence, if two stones, weighing a kilogramme, move each with a velocity of $3\frac{1}{2}$ kilometres per second directly towards one another, the energy liberated at the moment of impact is $2 \times \frac{1}{2} \times 10^3 (3\frac{1}{2} \times 10^5)^2$ or $12 \times 10^{13}$ ergs.

Now JOULE'S equivalent is $4{\cdot}2 \times 10^7$ ergs; hence, the energy liberated is about 3 million calories.

It is quite uncertain how much of each stone would be volatilised; but, if it were 3 grammes, there would be a million calories of energy applied to each gramme.

The melting temperature of iron is about 1500 degrees Centigrade, and the mean specific heat of iron may be about $\frac{1}{7}$.† Hence, about 300 calories are required to raise a gramme of iron from absolute zero to melting point. I do not know the latent heat of the melting of iron, but for platinum it is 27, and the latent heat of volatilisation of mercury is 62. Hence, about 400 or 500 calories suffice to raise a gramme of iron from absolute zero to volatilisation. It is clear, then, that there is energy enough, not only to volatilise the iron, but also to render the gas incandescent; and the same would be true if the mass operated on by the energy were 30 grammes instead of 3.

It must necessarily be obscure as to how a small mass of solid matter *can* take up a very large amount of energy in a small fraction of a second, but spectroscopic evidence seems to show that it does so; and, if so, we have what is virtually a violent explosive introduced between the two stones.

* If $v$ be the velocity of mean square, $v\sqrt{2}$ is the square root of the mean square of relative velocity.

† 'Physikalisch-Chemische Tabellen.' LANDOLT and BÖRNSTEIN.

In a direct collision each stone is probably shattered into fragments, like the splashes of lead when a bullet hits an iron target. But direct collision must be a comparatively rare event. In glancing collisions the velocity of neither body is wholly arrested, the concentration of energy is not so enormous (although probably still sufficient to effect volatilisation), and, since the stones rub past one another, more time is allowed for the matter round the point of contact to take up the energy; thus, the whole process of collision is much more intelligible. The nearest terrestrial analogy is when a cannon-ball bounds off the sea. In glancing collisions fracture will probably not be very frequent.

From these arguments, it is probable that, when two meteorites meet, they attain an effective elasticity of a high order of perfection; but there is, of course, some loss of energy at each collision. [It must, however, be admitted that on collision the deflection of path is rarely through a very large angle. But a succession of glancing collisions would be capable of reversing the path; and, thus, the kinetic theory of meteorites may be taken as not differing materially from that of gases.*]

Perhaps the most serious difficulty in the whole theory arises from the fractures which must often occur. If they happen with great frequency, it would seem as if the whole swarm of meteorites would degrade into dust. We know, however, that meteorites of considerable size fall upon the Earth; and, unless Mr. LOCKYER has misinterpreted the spectroscopic evidence, the nebulæ do now consist of meteorites. Hence, it would seem as if fracture was not of very frequent occurrence. It is easy to see that, if two bodies meet with a given velocity, the chance of fracture is much greater if they are large, and it is possible that the process of breaking up will go on only until a certain size, dependent on the velocity of agitation, is reached, and will then become comparatively unimportant.

When the volatilised gases cool, they will condense into a metallic rain, and this may fuse with old meteorites whose surfaces are molten. A meteorite in that condition will certainly also pick up dust. Thus, there are processes in action tending to counteract subdivision by fracture and volatilisation. The mean size of meteorites probably depends on the balance between these opposite tendencies. If this is so, there will be some fractures, and some fusions, but the mean mass will change very slowly with the mean kinetic energy of agitation. This view is, at any rate, adopted in the paper as a working hypothesis. It was not, however, possible to take account of fracture and fusion in the mathematical investigation, but the meteorites are treated as being of invariable mass.

## § 2. *On the Velocity of Agitation of Meteorites, and on its Secular Change.*

The velocity with which the meteorites move is derived from their fall from a great distance towards a centre of aggregation. In other words, the potential energy of their mutual attraction when widely dispersed becomes converted, at least partially,

* Added Nov. 16, 1888.

into kinetic energy. When the condensation of a swarm is just beginning, the mass of the aggregation towards which the meteorites fall is small; and, thus, the new bodies arrive at the aggregation with small velocity. Hence, initially, the kinetic energy is small, and the volume of the sphere within which hydrostatic ideas are (if anywhere) applicable is also small.

As more and more meteorites fall in that volume is enlarged, and the velocity with which they reach the aggregation is increased. Finally, the supply of meteorites in that part of space begins to fail, and the imperfect elasticity of the colliding bodies brings about a gradual contraction of the swarm.

I do not now attempt to trace the whole history of a swarm, but the object of the paper is to examine its mechanical condition at an epoch when the supply of meteorites from outside has stopped, and when the velocities of agitation and distribution of meteorites in space have arranged themselves into a sub-permanent condition, only affected by secular changes. This examination will enable us to understand, at least roughly, the secular change in the velocity and in the distribution of the meteorites as the swarm contracts, and will throw light on other questions.

### § 3. *Formulæ for Mean Square of Velocity, Mean Free Path, and Interval between Collisions.*

We have to investigate whether, when the solar system consisted of a swarm of meteorites, the velocities and encounters could have been such that the mechanics of the system can be treated as subject to the laws of hydrodynamics. The formulæ which form the basis of this discussion will now be considered.

For the sake of simplicity, the meteorites will, in the first instance, be treated as spheres of uniform size.

The sum of the masses of the meteorites is equal to that of the Sun, for the planets only contribute a negligible mass. If $M_0$ be the Sun's mass, and $m$ that of a meteorite, their number is $M_0/m$.

If, at each encounter between two meteorites, there were no loss of energy, the sum of the kinetic energies of all the meteorites would be equal to the potential energy lost in the concentration of the swarm from a condition of infinite dispersion, until it possessed its actual arrangement. In such a computation the rotational energy of the system is negligible.

Suppose the Sun's mass to be concentrated from infinite dispersion until it is arranged in the form of a homogeneous sphere of radius $a$ and density $\rho$. Then let the sphere be cut up into as many equal spaces as there are meteorites, and let the matter in each space be concentrated into a meteorite. When the number of meteorites is large, the potential energy lost in the first process is very great compared with that lost in the subsequent partial condensation into meteorites.* Thus, the energy lost in the partial condensation is negligible.

* It depends, in fact, on the square of the ratio of the diameter $2a$ to the linear dimension of one of the equal spaces.

If $\mu$ be the attractional constant, the lost energy of condensation is well known to be $\frac{3}{5}\mu M_0^2/a$. But on the hypothesis that there is no loss of energy at each encounter, this must be equal to the sum of the kinetic energies of all the meteorites. If, therefore, $v^2$ be the mean square of velocity of a meteorite, we must have $\frac{1}{2} M_0 v^2 = \frac{3}{5}\mu M_0^2/a$, so that $v^2 = \frac{6}{5}\mu M_0/a$.

But homogeneity of density and uniformity of kinetic energy of agitation are impossible; for the meteor-swarm must be much condensed towards its centre, so that we have largely underestimated the lost potential energy of the system. Also, the velocity of agitation must decrease towards the outside, or else the swarm would extend to infinity. Besides this, the partial conversion of molar into molecular energy, which must take place on each encounter, has been neglected.

We shall see below reason for believing that throughout a large central volume the mean square of velocity of agitation is nearly uniform, and that outside of this region it falls off.

Suppose, then, that $M$ is the mass and $a$ the radius of that portion of the swarm in which the square of velocity of agitation is uniform; let $v_0^2$ be that square of velocity, and let it be defined by reference to the potential of $M$ at distance $a$, so that

$$v_0^2 = \beta^2 \frac{\mu M}{a}, \qquad (1)$$

where $\beta$ is a coefficient for which a numerical value will be found below.

The square of velocity of agitation outside of the radius $a$ is to be denoted by $v^2$, and subsequent investigation will be necessary to evaluate $v^2$ in terms of $v_0^2$.

If we denote by $a_0$ the Earth's distance from the Sun, and by $u_0$ the Earth's velocity in its orbit, we have

$$u_0^2 = \mu \frac{M_0}{a_0}. \qquad (2)$$

Whence,

$$v_0 = \beta u_0 \left(\frac{M a_0}{M_0 a}\right)^{\frac{1}{2}}. \qquad (3)$$

If in any distribution of meteorites $w$ is the sum of the masses of all the meteorites in unit volume, or the density of the swarm at any point, and if $\lambda$ be that distance which is called in the kinetic theory of gases "the mean distance between neighbouring molecules," we have

$$\lambda^3 = \frac{m}{w}. \qquad (4)$$

Now, the mean density of that part of the swarm in which the kinetic energy of agitation is constant being $\rho$, we have

$$\rho = \frac{3M}{4\pi a^3}, \qquad (5)$$

and

$$\lambda^3 = 4\,\pi\, a^3 . \frac{m}{M} . \frac{\frac{1}{3}\rho}{w}. \qquad (6)$$

Suppose that $s$ is "the radius of the sphere of action" of a meteorite, so that when two of them approach so that the distance between their centres is $s$ there is a collision.

Let $L$ and $T$ be the mean free path and mean interval between collisions. Then, since the mean velocity is $v\sqrt{(8/3\pi)}$, we have, according to the kinetic theory of gases,*

$$L = \frac{\lambda^3}{\pi s^2 \sqrt{2}}, \qquad T = \frac{L}{v}\sqrt{\tfrac{3}{8}}\,\pi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Then, on substitution from (4), (5), and (6), we have

$$L = \left[\frac{a_0^3\sqrt{8}}{M_0}\right]\left(\frac{a}{a_0}\right)^3 \frac{M_0}{M}\cdot\frac{\frac{1}{3}\rho}{w}\cdot\frac{m}{s^2}, \qquad T = \left[\frac{a_0^3\sqrt{3\pi}}{M_0 u_0}\right]\left(\frac{a}{a_0}\right)^{\frac{7}{2}}\cdot\frac{1}{\beta}\left(\frac{M_0}{M}\right)^{\frac{3}{2}}\frac{v_0}{v}\cdot\frac{\frac{1}{3}\rho}{w}\cdot\frac{m}{s^2}. \quad . \quad (8)$$

Now let

$$\left.\begin{aligned} u_0 &= \left(\frac{\mu M_0}{a_0}\right)^{\frac{1}{2}}, & u &= u_0\left(\frac{a_0}{a}\right)^{\frac{1}{2}} \\ l_0 &= \frac{a_0^3\sqrt{8}}{M_0}, & l &= l_0\left(\frac{a}{a_0}\right)^3 \\ \tau_0 &= \frac{a_0^3\sqrt{3\pi}}{M_0 u_0}, & \tau &= \tau_0\left(\frac{a}{a_0}\right)^{\frac{7}{2}} \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and we have

$$\left.\begin{aligned} v &= \beta\, u\left(\frac{M}{M_0}\right)^{\frac{1}{2}} \\ L &= l\,.\left(\frac{M_0}{M}\right)\cdot\frac{\frac{1}{3}\rho}{w}\cdot\frac{m}{s^2} \\ T &= \tau\,.\frac{1}{\beta}\left(\frac{M_0}{M}\right)^{\frac{3}{2}}\cdot\frac{v_0}{v}\cdot\frac{\frac{1}{3}\rho}{w}\cdot\frac{m}{s^2} \end{aligned}\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

We now proceed to calculate $u_0$, $l_0$, $\tau_0$, and also $2a_0/l_0$, using the centimetre-gramme-second system of units.

The Sun's mass may be taken as 315,511 times that of the Earth, and the Earth as $6{\cdot}14 \times 10^{27}$ grammes†; hence

$$M_0 = 10^{33{\cdot}28718} = 1{\cdot}9372 \times 10^{33} \text{ grammes.}$$

The attractional constant and the Earth's mean distance from the Sun are

$$\mu = \frac{648}{10^{10}}, \qquad a_0 = 1{\cdot}487 \times 10^{13} \text{ cm.}$$

* MEYER, 'Kinetische Theorie der Gase.'

† Here and elsewhere I generally use EVERETT'S 'Units and Physical Constants.'

With these values

$$\left.\begin{aligned} u_0 &= 10^{6\cdot46323} = 2{,}905{,}600 \text{ cm. per sec.} \\ l_0 &= 10^{6\cdot68129} = 4{,}800{,}600 \text{ cm.} \\ \tau_0 &= 10^{0\cdot25366} = 1\cdot79334 \text{ sec.} \\ \frac{2a_0}{l_0} &= 10^{6\cdot79204} = 6{,}195{,}000 \end{aligned}\right\} \quad \ldots \ldots \quad (11)$$

The dimensions of $l_0$ and $\tau_0$ are not those of length and time; but, if meteorites of 1 gramme mass, with sphere of action 1 centimetre, and "velocity of mean square" of agitation equal to the Earth's velocity in its orbit, have density of distribution equal to one-third of the mean density of the sphere $M$, then $l_0$, $\tau_0$ will be the mean free path and time, as stated in centimetres and seconds. We may thus regard $l_0$, $\tau_0$ as a length and time, provided care be taken in the subsequent use of the symbols to adhere to the c.g.s. system of units.

### § 4. *On the Equilibrium of a Gas at Uniform Temperature in Concentric Spherical Layers under its own Gravitation.*

It is assumed provisionally that the conditions are satisfied which permit us to regard the swarm of meteorites as a quasi-gas, subject to the laws of hydrostatics.

The solution of this problem, then, becomes a necessary preliminary to the discussion of the kinetic theory of meteorites. The equilibrium of a gas under its own gravitation has been ably discussed by Professor RITTER in one of his series of papers on gaseous planets.* The intrinsic interest of the problem renders an independent solution valuable. Suppose, then, that a mass $M_1$ of gas is enclosed in a spherical envelope of radius $a_1$, and is in equilibrium in concentric spherical layers. Let $v_1^2$, the mean square of the velocity of agitation of the gaseous molecules, be defined by reference to the potential of the mass $M_1$ at the radius $a_1$, so that

$$v_1^2 = \beta_1^2 \frac{\mu M_1}{a_1},$$

where $\beta_1^2$ is a numerical coefficient, and $\mu$ is the attractional constant.

Let $p$ and $w$ be the pressure and density of the gas at radius $r$, and $k$ the modulus of elasticity, so that

$$p = k w,$$

$$k = \tfrac{1}{3} v_1^2 = \tfrac{1}{3} \beta_1^2 \frac{\mu M_1}{a_1}.$$

* "Untersuchungen über die Höhe der Atmosphäre und die Constitution gasförmiger Weltkörper," 'WIEDEMANN'S Annalen' (New Series), vol. 16, 1882, p. 166. A very elegant solution of part of my problem has also been given by Mr. G. W. HILL in the 'Annals of Mathematics,' vol. 4, No. 1, p. 19 (February, 1888). Mr. HILL'S paper only reached my hands after my own calculations had been completed, and I therefore adhere to my own less elegant method. Mr. HILL has obviously not seen M. RITTER'S papers.

Then the equation for the hydrostatic equilibrium of the gas is

$$\frac{r^2}{w}\frac{dp}{dr} + 4\pi\mu\int_0^r wr^2\,dr = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

It is obvious that $\frac{-r^2}{\mu w}\frac{dp}{dr}$ is equal to the whole mass enclosed inside radius $r$, and this relation will hold however the equation be transformed, provided we do not multiply the equation by any factor.

In consequence of the relation between $p$ and $w$ this may be written

$$k\left[r^2\frac{d}{dr}\log w + \frac{4\pi\mu}{k}\int_0^r wr^2\,dr\right] = 0.$$

If $\rho_1$ be the mean density of the mass $M_1$, we have

$$4\pi\mu = \frac{3\mu M_1}{\rho_1 a_1^3} = \frac{9k}{\beta_1^2 a_1^2 \rho_1}.$$

Hence, we may write the equation (12) in the form

$$ka_1\left[\frac{r^2}{a_1}\frac{d}{dr}\log w + \frac{9}{\beta_1^2}\int_0^r \frac{w}{\rho_1}\frac{r^2}{a_1^3}\,dr\right] = 0.$$

Now, let

$$x_1 = \frac{a_1}{r}, \qquad \frac{w}{\rho_1} = \tfrac{1}{9}\beta_1^2 e^{y_1},$$

and the equation becomes

$$\tfrac{1}{3}\beta_1^2\mu M_1\left[-\frac{dy_1}{dx_1} + \int_{x_1}^{\infty}\frac{e^{y_1}}{x_1^4}\,dx_1\right] = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

By differentiation we obtain the equation

$$\frac{d^2y_1}{dx_1^2} + \frac{e^{y_1}}{x_1^4} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

It is obvious from (13) that $\frac{1}{3}\beta_1^2 M_1\,dy_1/dx_1$ is the mass enclosed inside radius $a/x_1$, and therefore $\frac{1}{3}\beta_1^2\,dy_1/dx_1$ is equal to unity when $x = 1$.

A general analytical solution of (14) does not seem to be attainable, and recourse must be had to numerical processes. Although this is an equation of the second degree, and its general solution must involve two arbitrary constants, we shall see (as pointed out by M. RITTER) that the general solution, as applicable to our problem, may be deduced from one single numerical solution. M. RITTER proceeds by a graphical method, which he has worked with surprising accuracy. I shall therefore adopt an analytical and numerical method, which, although laborious, is susceptible of greater accuracy.





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Whatever be the arrangement of the gas, the density at the centre must have some value. I therefore start with a central density $\omega$, corresponding to the value $\eta$ of $y_1$, so that

$$\frac{\omega}{\rho_1} = \tfrac{1}{9}\beta_1^2 e^{\eta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

For the sake of brevity the suffixes 1 will be now omitted from the various symbols, to be reaffixed later when they are required.

At the centre, where $x$ is infinite, $dy/dx$, $d^2y/dx^2$, &c., are all zero, and we put $y = \eta$.

Let $\xi = e^{\eta}/x^2$, and let us assume

$$\begin{aligned} y &= \eta + v \\ &= \eta - A_1\xi + A_2\xi^2 - A_3\xi^3 + \dots \; . \end{aligned}$$

Now, the differential equation (14) to be satisfied is

$$x^2\frac{d^2y}{dx^2} = -\frac{e^y}{x^2} = -\xi e^v.$$

But

$$x^2\frac{d^2y}{dx^2} = -2\,.\,3A_1\xi + 4\,.\,5A_2\xi^2 - 6\,.\,7A_3\xi^3 + \dots\;,$$

and by expanding $e^v$ we obtain

$$\begin{aligned} -\xi e^v = &-\xi + A_1\xi^2 - (A_2 + \tfrac{1}{2}A_1^2)\,\xi^3 + (A_3 + A_1A_2 + \tfrac{1}{2.3}A_1^3)\,\xi^4 \\ &- (A_4 + A_1A_3 + \tfrac{1}{2}A_2^2 + \tfrac{1}{2}A_1^2A_2 + \tfrac{1}{2.3.4}A_1^4)\,\xi^5 \\ &+ (A_5 + A_1A_4 + A_2A_3 + \tfrac{1}{2}A_1^2A_3 + \tfrac{1}{2}A_1A_2^2 + \tfrac{1}{2.3}A_1^3A_2 + \tfrac{1}{2.3.4.5}A_1^5)\,\xi^6 - \dots \end{aligned}$$

By equating coefficients in these two series, I find

$$A_1 = \tfrac{1}{6}, \qquad A_2 = \tfrac{1}{120}, \qquad A_3 = \tfrac{1}{1890}, \qquad A_4 = \tfrac{61}{1,632,960},$$
$$A_5 = \tfrac{629}{224,532,000}, \qquad A_6 = \tfrac{34\cdot07383\dots}{156\times10^6}, \quad \text{\&c.},$$

and

$$\log A_1 = 9\cdot2218487, \qquad \log A_2 = 7\cdot9208188, \qquad \log A_3 = 6\cdot7235382,$$
$$\log A_4 = 5\cdot5723543, \qquad \log A_5 = 4\cdot4473723, \qquad \log A_6 = 3\cdot3392964\,;$$

whence, by extrapolation,

$$\log A_7 = 2\cdot243, \qquad \log A_8 = 1\cdot13.$$

In M. RITTER's paper, already referred to, he takes a certain function $u$ as equal to 1·031, when the radius is unity. Now, RITTER's function $u$ is equal in my notation to

$\frac{1}{2}(w/\rho) \div \frac{1}{9}\beta^2 a^2/r^2$ or $\frac{1}{2}e^y/x^2$. It follows, therefore, that RITTER takes the surface value of $y = \log_e 2{\cdot}062$. But he intends the central density to be 100 times the surface density; hence, to take the same solution, we must have $e^\eta$ equal to 100 times 2·062. Therefore, his value of $\eta$ would be

$$\eta = \log_e 100 + \log_e 2{\cdot}062,$$

or

$$\eta = 5{\cdot}3288465.$$

Now, as I want to make a comparison between my solution and his, I start with this value of $\eta$. The only object attained by the choice of this particular value is that the two solutions become easily comparable. It will be seen below that the value of $\eta$ does not make the central density exactly 100 times the surface density, but only satisfies that condition approximately. In RITTER's graphical treatment of the problem this value 100 is the exact datum, whereas in my method we start with an exact value of $\eta$, and proceed to find the ratio of central to surface density.

With this value of $\eta$ (whence $\log_{10} e^\eta = 2{\cdot}3142888$) I find the following series for $y$:—

$$y = 5{\cdot}3288465 - \frac{34{\cdot}3667}{x^2} + \frac{354{\cdot}321}{x^4} - \frac{4638{\cdot}79}{x^6} + \frac{67{,}532{\cdot}0}{x^8} - \frac{1{,}044{,}280}{x^{10}} + \frac{16{,}789{,}000}{x^{12}} - \frac{2{\cdot}77 \times 10^8}{x^{14}} + \frac{4{\cdot}4 \times 10^9}{x^{16}} - \ldots; \quad . \quad . \quad (16)$$

and, by differentiation, the series for $dy/dx$ is obvious.

This series will be very accurate from $x = \infty$ to about $x = 8$. Thus, when $r/a = {\cdot}1$, or $x = 10$, we have

$$y = 5{\cdot}016558, \qquad \frac{dy}{dx} = +{\cdot}0568910,$$

and even the far less convergent series for $d^2y/dx^2$ gives $-{\cdot}0150891$, agreeing with $-e^y/x^4$ to the last place of decimals. When $r/a = {\cdot}125$, or $x = 8$, we have *

$$y = 4{\cdot}863925, \qquad \frac{dy}{dx} = {\cdot}101168,$$

whence,

$$\frac{d^2y}{dx^2} = -{\cdot}031624,$$

with $y$ correct to four, and probably to five, places of decimals, and $dy/dx$ probably correct to four places of decimals. This is amply sufficient for our purpose. Indeed, accuracy of this order would be altogether pedantic, were it not that the errors accumulate.

* Even when $x = 5$, I find from this series $y = 4{\cdot}342$, which lies very near to $y = 4{\cdot}332$, found below. But the series for $dy/dx$ is useless.

We cannot, then, rely on this method of procedure beyond the region included between $x = \infty$ and $x = 8$, and must now make a new departure.

Since

$$\frac{d^2y}{dx^2} = -\frac{e^y}{x^4},$$

$$\log\left(-\frac{d^2y}{dx^2}\right) = y - 4\log x\,;$$

therefore,

$$\frac{d^3y}{dx^3} = \frac{d^2y}{dx^2}\left(\frac{dy}{dx} - \frac{4}{x}\right). \quad \text{. . . . . . . . . . (17)}$$

Now, let

$$A_n = \frac{1}{n!}\frac{d^ny}{dx^n},$$

where, after differentiation, $x$ is put equal to $c$, a constant.

Then (17) may be written—

$$A_3 = \frac{2!}{3!}A_2\left(A_1 - \frac{4}{c}\right). \quad \text{. . . . . . . . . . (18)}$$

Now, it is clear that

$$\frac{d^pA_n}{dc^p} = \frac{\overline{n+p}!}{n!}A_{n+p}$$

Hence, differentiating (18) $n - 3$ times, we have

$$\frac{n!}{3!}A_n = \sum_{q=0}^{q=n-3} \frac{\overline{n-3}!}{\overline{n-3-q}!\,q!}\,\frac{2!}{3!}\,\frac{\overline{n-q-1}!}{2!}A_{n-q-1}\left\{\frac{\overline{q+1}!}{1!}A_{q+1} + (-)^q\frac{4\,(q!)}{c^{q+1}}\right\},$$

or

$$A_n = \frac{1}{n\,.\,\overline{n-1}\,.\,\overline{n-3}}\sum_{q=0}^{q=n-3}(n-q-1)(n-q-2)A_{n-q-1}\left\{(q+1)A_{q+1} + (-)^q\frac{4}{c^{q+1}}\right\},$$

or

$$A_n = \frac{1}{n\,.\,\overline{n-1}\,.\,\overline{n-3}}\left\{2\,.\,1A_2\left[(n-2)A_{n-2} + (-)^n\frac{4}{c^{n-2}}\right]\right.$$
$$\left. + 3\,.\,2A_3\left[(n-3)A_{n-3} - (-)^n\frac{4}{c^{n-3}}\right] + \ldots\right\}. \quad \text{(19)}$$

Now, if, for a given value of $x$, viz., $c$, we know $y$, or $A_0$, and $dy/dx$, or $A_1$, then we can compute $A_2$ from the formula $-\frac{1}{2}e^{A_0}c^{-4}$; and, by the formula (18), viz. :—

$$A_3 = \frac{1}{3\,.\,2\,.\,1}\left\{2\,.\,1A_2\left[A_1 - \frac{4}{c}\right]\right\},$$

$A_3$ may be computed.

Afterwards, $A_4$, $A_5$, &c., may be computed by successive applications of (19). This being so,

$$y = A_0 + A_1(x-c) + A_2(x-c)^2 + \ldots, \quad \ldots \quad (20)$$

$$\frac{dy}{dx} = A_1 + 2A_2(x-c) + 3A_3(x-c)^2 + \ldots .$$

In these series $x$ may have any value, provided the series converges adequately. The convergence may be much improved by an artifice, which, however, I unfortunately did not discover until most of the computations were completed. Let us add and subtract $\log 2x^2$ on the right-hand side of (20).

Now,

$$\log_e 2x^2 = \log_e 2c^2 + 2\log_e\left[1 + \frac{x-c}{c}\right]$$

$$= \log_e 2c^2 + 2\frac{x-c}{c} - \tfrac{2}{2}\frac{(x-c)^2}{c^2} + \tfrac{2}{3}\frac{(x-c)^3}{c^3} - \ldots .$$

If, then, we write

$$B_0 = A_0 - \log_e 2c^2, \qquad B_1 = A_1 - \frac{2}{c}, \qquad B_2 = A_2 + \frac{2}{2c^2}, \qquad B_3 = A_3 - \frac{2}{3c^3} \text{ \&c.},$$

we have

$$y = \log_e 2x^2 + B_0 + B_1(x-c) + B_2(x-c)^2 + \ldots, \quad \ldots \quad (21)$$

a more convergent series than that with the $A$'s.

The simplest way of computing the $B$'s appears to be by first computing the $A$'s.

The process for obtaining the numerical solution is then as follows:—

We have the values of $y$, $dy/dx$, $\frac{1}{2}dy^2/dx^2$ when $x = 8$, that is to say, of $A_0$, $A_1$, $A_2$ when $c = 8$. From these the successive $A$'s and $B$'s are computed, and the resulting series gives the values of $y$ and $dy/dx$ when $x$ is 5 or $r = \cdot 2$. Starting from this point a new series gives the result when $r = \cdot 3$, another series gives the values for $r = \cdot 4$, and so on. Later in the calculation several values may be computed from one formula.*

When the computation has been carried out to $r = a$, we have reached the spherical envelope, but that envelope may be replaced by another at any more remote distance from the centre. Thus, the integration may be pursued for values of $x$ less than unity, and when the lower limit is zero the envelope is at infinity.

If we write

$$\log u = B_0 + B_1(x-c) + B_2(x-c)^2 + \ldots,$$

* If the series be carried as far as $B_8$, several steps may be included in one series. For example, the first series, when $c = 8$, may be pushed even as far as $r = \cdot 4$ without serious error; for it gives $y = 2\cdot 960$ instead of the true value $2\cdot 965$, and $dy/dx = \cdot 957$, instead of the true value $\cdot 944$. I have not, however, been satisfied with this degree of accuracy.

we have

$$e^{y} = u \, . \, 2x^2,$$

and

$$\frac{w}{\rho} = \frac{2}{9}\frac{\beta^2 a^2}{r^2} \cdot u.$$

But it may easily be seen that $2\beta^2a^2/9r^2$ is a particular solution of the problem; hence, $u$ is a factor by which the particular solution is to be multiplied to obtain the general solution. The function $u$ is given by

$$u = \tfrac{1}{2}\frac{e^{y}}{x^2} = -\tfrac{1}{2}x^2\frac{d^2y}{dx^2} \cdot \qquad (22)$$

A table of the values of $u$ is given below, showing how the general solution shades off into the particular solution. This function, $u$, is also tabulated by RITTER, and I made use of its value, when $x = 1$, to determine the value of $\eta$, with which the integration is to begin. I find, however (see Table I.), that, when $x = 1$, $u = 1{\cdot}0063$, in place of $1{\cdot}031$, as given by him.

The last row in Table I. gives the ratio of the central density $\omega$ to $w$, the density at the distance $r$; this ratio is equal to $e^{\eta - y_1}$.

The following Table gives the results of the computation, and the suffix 1 is reintroduced in the several symbols.

I.—Table of Results.

| $\frac{1}{x_1} = \frac{r}{a_1} =$ | 0 | ·1 | ·125 | ·2 | ·3 | ·4 | ·5 | ·6 | ·7 | ·8 | ·9 | 1·0 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $y_1 =$ | 5·328845 | 5·01656 | 4·8639 | 4·3317 | 3·6064 | 2·9653 | 2·4235 | 1·9672 | 1·5794 | 1·2458 | ·9553 | ·6995 |
| $\frac{dy_1}{dx_1} =$ | 0·0 | ·05689 | ·10117 | ·2967 | ·6217 | ·9442 | 1·2404 | 1·5098 | 1·7569 | 1·9867 | 2·2030 | 2·4087 |
| $\frac{d^2y_1}{dx_1^2} =$ | 0·0 | −·01509 | −·03162 | −·1217 | −·2984 | −·4967 | −·7053 | −·9267 | −1·1647 | −1·4237 | −1·7055 | −2·0126 |
| $u = -\frac{1}{2}x_1^2 \frac{d^2y_1}{dx_1^2} =$ | 0·0 | ·75445 | 1·0120 | 1·5215 | 1·6577 | 1·5521 | 1·4107 | 1·2871 | 1·1888 | 1·1123 | 1·0528 | 1·0063 |
| $\frac{\omega}{w} = e^{\eta - y_1} =$ | 1·0000 | 1·3666 | .. | 2·7105 | 5·598 | 10·63 | 18·29 | 28·84 | 42·50 | 59·32 | 79·33 | 102·45 |

| $\frac{1}{x_1} = \frac{r}{a_1} =$ | 1·0 | 1·1 | 1·2 | 1·5 | 2·0 | 2·5 | 3·0 | $\infty \left(\frac{1}{x_1}\right)$ |
|---|---|---|---|---|---|---|---|---|
| $y_1 =$ | ·6995 | ·4717 | ·2672 | −·241 | −·864 | −1·328 | −1·699 | $-\infty$ $(\log 2x_1^2)$ |
| $\frac{dy_1}{dx_1} =$ | 2·4087 | 2·6062 | 2·7971 | 3·342 | 4·202 | 5·139 | 6·038 | $+\infty \left(\frac{2}{x_1}\right)$ |
| $\frac{d^2y_1}{dx_1^2} =$ | −2·0126 | −2·3466 | −2·7089 | −3·976 | −6·744 | −10·35 | −14·82 | $-\infty \left(-\frac{2}{x_1^2}\right)$ |
| $u = -\frac{1}{2}x_1^2 \frac{d^2y_1}{dx_1^2} =$ | 1·0063 | ·9697 | ·9406 | ·884 | ·843 | ·828 | ·823 | 1·0 |
| $\frac{\omega}{w} = e^{\eta - y_1} =$ | 102·45 | 128·7 | 157·8 | 263 | 489 | 778 | 1127 | $\infty \left(\frac{1}{2}\frac{e^{\eta}}{x_1^2}\right)$ |

It will be noticed that $u$ rises from zero to a maximum of about 1·66, falls to a minimum of about ·82, and then rises to unity.

Since $\frac{1}{3}\beta_1^2\, dy_1/dx_1 = 1$ when $x_1 = 1$, we have $\frac{1}{3}\beta_1^2 = 1/2{\cdot}4087 = {\cdot}4152$.

M. RITTER has ·4143 for this constant, which he calls $m$.

It appears from the Table that the density at the centre is $102\frac{1}{2}$ times as great as that where $r = a_1$. M. RITTER'S solution is intended to make that ratio exactly 100, but this solution shows that we ought to have started with a slightly different value of $\eta$ to obtain that result.

In the general solution of the differential equation $d^2y/dx^2 = -e^y/x^4$ the two arbitrary constants may be taken to be the values of $y$ and $dy/dx$ when $x$ is infinite. Now, we have taken arbitrarily $y = 5{\cdot}329$ when $x$ is infinite, and the physical conditions of the problem imply that $dy/dx$ is zero when $x$ is infinite. For if $dy/dx$ had any positive or negative value different from zero, it would mean that at the centre there was a nucleus of infinitely small dimensions, but of finite positive or negative mass. Now, $a_1$ is that distance from the centre at which the density is 1/102·45 of the central density; hence, we may regard $a_1$ as the arbitrary constant of the solution. Whatever be the elasticity of the gas, we may always take as our unit of length that distance from the centre of the nebula at which the density has fallen to 1/102·45 of its central value. Hence, the above table gives the general solution of the problem, subject, however, to the condition that there is no central nucleus.

If we view the nebula from a very great distance, $a_1$ appears very small, and thus the solution of the problem becomes $y = \log 2x^2$. It is easy to verify that this is a particular algebraic solution of the differential equation, as is pointed out by RITTER in his paper.* I found this solution very useful in a preliminary consideration of the problem treated in this paper.

The next point which we have to consider is the form which the solution will take, if, instead of taking $a_1$ as the unit of length, we take any other value.

The density at any distance and the elasticity are to remain unchanged, but are to be referred to new constants.

Thus, $w$, $r$, $v^2$ remain unchanged, but are to be referred to $M$, $\rho$, $\beta^2$, $a$, instead of to $M_1$, $\rho_1$, $\beta_1^2$, $a_1$.

Now, since $w$ remains unchanged,

$$\tfrac{1}{9}\beta^2\rho\, e^y = \tfrac{1}{9}\beta_1^2\rho_1 e^{y_1},$$

and, since $v^2$ remains unchanged,

$$\beta^2\rho\, a^2 = \beta_1^2\rho_1 a_1^2.$$

Also

$$x = \frac{a}{r} = x_1\frac{a}{a_1}\,.$$

* I have made use of this solution in a paper in the 'Proceedings of the Royal Society,' Dec. 3, 1883, and it has also been referred to in a paper by Sir W. THOMSON, 'Phil. Mag.,' vol. 23, p. 287. Sir W. THOMSON'S paper covers much the same ground as some of M. RITTER'S earlier papers, but was written by him independently and in ignorance of them.

From these relations it is clear that

$$y = y_1 - 2 \log \frac{a_1}{a},$$

and

$$M = \tfrac{1}{3} \beta_1^2 M_1 \frac{dy_1}{dx_1}\left(x_1 = \frac{a_1}{a}\right).$$

Then, since $\frac{1}{3} \beta^2 \, dy/dx = 1$, when $x = 1$, and since $dy = dy_1$ and $dx = dx_1 \, a/a_1$, it follows that

$$\tfrac{1}{3} \beta^2 = \frac{1}{x_1 \, dy_1/dx_1}, \text{ when } x_1 = \frac{a_1}{a}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

This relationship has been already used for determining $\beta_1^2$.

It is obvious also that

$$\frac{\rho}{\rho_1} = \frac{x_1^3 \, dy_1/dx_1, \text{ when } x_1 = a_1/a}{x_1^3 \, dy_1/dx_1, \text{ when } x_1 = 1}.$$

Therefore,

$$\frac{w}{\rho} = \frac{\frac{1}{3} e^{y_1}}{x_1^3 \, dy_1/dx_1, \text{ when } x_1 = a_1/a}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

If $w_0$ be the density when $r = a$, we have

$$\frac{w_0}{\rho} = \frac{\frac{1}{3} e^{y_1}, \text{ when } x_1 = a_1/a}{x_1^3 \, dy_1/dx_1} = -\tfrac{1}{3} \cdot \frac{x_1 \, d^2y_1/dx_1^2}{dy_1/dx_1}, \text{ when } x_1 = a_1/a. \quad . \quad (25)$$

If $p_0$ be the pressure when $r = a$, we have

$$p_0 = \tfrac{1}{3} v^2 w_0 = \tfrac{4}{9} \pi \mu a^2 \rho^2 \, . \, \beta^2 \frac{w_0}{\rho}.$$

If, therefore, we write $P = \frac{4}{3} \pi \mu a^2 \rho^2$,

$$\frac{p_0}{P} = \tfrac{1}{3} \beta^2 \, . \, \frac{w_0}{\rho} = -\frac{d^2y_1/dx_1^2}{(dy_1/dx_1)^2}, \text{ when } x_1 = a_1/a. \quad . \quad . \quad . \quad . \quad (26)$$

By (26) we are able to find how the pressure on an envelope of given radius $a$ varies with the variation of the temperature of a given mass $M$ of gas contained in it. By means of the formulæ (23), (25), (26), we are now able to obtain from the original solution any number of other ones; for, after the changes have been effected in the notation, we may proceed to magnify or diminish all the various values of $a$ until they are of one size, and we shall thus obtain the solution for a gas at any temperature whatever.

I shall now proceed to give a table of results when the standard radius $a$, which may be conveniently called the boundary, is placed successively infinitely near the centre, where $r = 0 \times a_1$, at $r = \cdot 1 \times a_1$, $r = \cdot 2 \times a_1$, and so on. The first line of entries gives the various values of $\frac{1}{3}\beta^2$ (computed from (23)), on which the elasticity of the gas depends; the second line gives $w_0/\rho$ (computed from (25)), or the ratio of the boundary density to the mean density of all inside of it; the third line gives $p_0/P$ (computed from (26)), by which we trace the variations of pressure at the boundary.

TABLE II.

| Value of $a$ by reference to former solution $\frac{a}{a_1} =$ | 0 | ·1 | ·2 | ·3 | ·4 | ·5 | ·6 | ·6264 | ·7 | ·8 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\frac{\frac{1}{3}v^2}{\mu M/a} = \frac{1}{3}\beta^2\left[= \frac{1}{x_1\ dy_1/dx_1}\right] =$ | ∞ | 1·7577 | ·6741 | ·4826 | ·4236 | ·4031 | ·3974 | ·3972 | ·3984 | ·4027 |
| $\frac{w_0}{\rho}\left[= \frac{-\frac{1}{3}x_1\ d^2y_1/dx_1^2}{dy_1/dx_1}\right] =$ | 1·0000 | ·8841 | ·6838 | ·5333 | ·4383 | ·3791 | ·3410 | $\frac{1}{3}$ | ·3158 | ·2986 |
| $\frac{p_0}{P}\left[= \frac{-d^2y_1/dx_1^2}{[dy_1/dx_1]^2}\right] =$ | ∞ | 4·662 | 1·383 | ·772 | ·557 | ·458 | ·407 | ·397 | ·377 | ·361 |

| Value of $a$ by reference to former solution $\frac{a}{a_1} =$ | ·9 | 1·0 | 1·25 | 1·5 | 2·0 | 2·5 | 3·0 | ∞ |
|---|---|---|---|---|---|---|---|---|
| $\frac{\frac{1}{3}v^2}{\mu M/a} = \frac{1}{3}\beta^2\left[= \frac{1}{x_1\ dy_1/dx_1}\right] =$ | ·4085 | ·4152 | ·4325 | ·449 | ·476 | ·487 | ·497 | $\frac{1}{2}$ |
| $\frac{w_0}{\rho}\left[= \frac{-\frac{1}{3}x_1\ d^2y_1/dx_1^2}{dy_1/dx_1}\right] =$ | ·2867 | ·2785 | ·2676 | ·264 | ·267 | ·269 | ·273 | $\frac{1}{3}$ |
| $\frac{p_0}{P}\left[= \frac{-d^2y_1/dx_1^2}{[dy_1/dx_1]^2}\right] =$ | ·351 | ·347 | ·347 | ·356 | ·382 | ·392 | ·406 | $\frac{1}{2}$ |

The minimum value of $w_0/\rho$ occurs when $a/a_1 = 1·6$ very nearly, for, when $a/a_1 = 1·4$, $1·5$, $1·6$, I find $w_0/\rho = ·26521$, $·26437$, $·26425$ respectively.* When $r/a_1 = 1·6$, $y_1 = -·38435$ and $dy_1/dx_1 = 3·5180$. The minimum value of $p_0/P$ occurs when $a/a_1 = 1·1$ very nearly, for, when $a/a_1 = 1·0$, $1·1$, $1·2$, I find $p_0/P = ·3469$ $·3455$, $·3462$ respectively.

When $w_0/\rho$ is a minimum, the density at the centre is 381 times that at the boundary, and, when $p_0/P$ is a minimum, the density at the centre is 129 times that at the boundary. M. RITTER finds the pressure to be a minimum when this ratio is 258, instead of 129. As this corresponds to $a/a_1 = 1·5$, this discrepancy between our solutions is not so large as might be expected from the great discrepancy between these results, and I cannot but think that my result is more accurate than his.

The minimum value of $\frac{1}{3}\beta^2$ occurs when $a/a_1 = ·6264$, and its value is $·39723$. This value makes the surface density exactly one-third of the mean density, for $\frac{1}{3}\beta^2$ is a minimum when $x_1\ dy_1/dx_1$ is a maximum, and this occurs when $x_1 d^2y_1/dx_1^2 + dy_1/dx_1 = 0$; and, when this relationship is satisfied, $w_0/\rho = \frac{1}{3}$.

It is interesting to note that in this case $\beta^2$ is very nearly equal to $\frac{6}{5}$, so that the

* Mr. HILL finds that the minimum value of $w_0/\rho$ approximates to $\frac{4}{15}$, or ·2667. The agreement between our results is satisfactory.

total internal kinetic energy of agitation of the sphere of gas at minimum temperature limited by the radius $a$ is $\frac{1}{2}(\frac{6}{5}\mu M/a)M = \frac{3}{5}\mu M^2 a$ *very nearly*. Now, the energy lost in the concentration of a homogeneous sphere $M$ from a condition of infinite dispersion is *exactly* $\frac{3}{5}\mu M^2/a$. It might, therefore, be suspected that ·39723 is only an approximation to $\frac{2}{5}$, which may be the rigorous value. But my numerical calculations were carried out with so much care that I find it almost impossible to believe that there is an error as large as 3 in the third place of decimals, or, indeed, any error at all in the third figure. Moreover, it would be expected that, if this very simple relationship is rigorously correct, it would be possible to prove it rigorously, just as it is rigorously shown above that $w_0/\rho = \frac{1}{3}$; but I am unable to find any analytical relationships by which the minimum value of $\frac{1}{3}\beta^2$ can be deduced. If my arithmetical process be correctly carried out, then we ought to find that, when $r = \cdot 6264$, $dy_1/dx_1$ should be equal to $-x_1 d^2y_1/dx_1^2$ or $e^{y_1}/x_1^3$. Now, I find that, when $r = \cdot 6264$, $dy_1/dx_1 = 1\cdot 57703$ and $e^{y_1}/x_1^3 = 1\cdot 5770$, so that the two agree to four places of decimals. I conclude, therefore, that the true minimum of $\frac{1}{3}\beta^2$ is ·3972.*

It will be observed that, as $a/a_1$ increases to infinity, $\frac{1}{3}\beta^2$ terminates by being equal to $\frac{1}{2}$. M. RITTER has found that it rises above $\frac{1}{2}$, and oscillates about that value an indefinite number of times with diminishing amplitude, gradually settling down to $\frac{1}{2}$ as $a/a_1$ becomes infinite. The values in the preceding table are not, however, carried far enough to exhibit these oscillations of $\frac{1}{3}\beta^2$. A consequence of this result is that there are a number of modes of equilibrium of a gas at a given temperature, provided that the temperature lies within certain narrow limits. This very remarkable conclusion is rendered more intelligible by Mr. HILL's treatment than by M. RITTER's.

This point has, however, no bearing on the present investigation.

In any one of the solutions comprised in Table II. we may complete the table of densities by the formula (24), viz.,

$$\frac{w}{\rho} = \frac{\frac{1}{3}e^{y_1}}{x_1^3\, dy_1/dx_1\,(x_1 = a_1/a)},$$

and I shall later proceed to do this in the one case which has interest for our present problem, namely, where the temperature is a minimum, so that $a/a_1$ is ·6264. The full numerical results may be more conveniently given hereafter, and it will only be now necessary to indicate how they are to be computed.

When, for example, $r = \cdot 1 \times a_1$, $r/a = \cdot 1/\cdot 6264 = \cdot 1596$; thus, our equidistant values of the density and other functions will proceed by multiples of ·1596 $a$ up to ·9578 $a$, and the limit of the isothermal sphere is where $r = a$.

When the temperature is a minimum $\frac{1}{3}\beta^2 = \cdot 39723$, and we have $w_0 = \frac{1}{3}\rho$; therefore, $w/w_0 = w/\frac{1}{3}\rho$, and, therefore, if $y_{1,0}$ be the value of $y_1$, when $r = \cdot 6264\, a_1$,

* This is confirmed by Mr. HILL. His equation $s = z$ is equivalent to $x_1^2 d^2y_1/dx_1^2 + x_1 dy_1/dx_1 = 0$, and it appears from his tables that $s = z = 2\cdot 517$. Now, $s = 3/\beta^2$, and the reciprocal of $2\cdot 517$ is ·397.

$w/\frac{1}{3}\rho = e^{y_1 - y_{10}}$. Thus, for example, at the centre, $w/\frac{1}{3}\rho$ is 32·14, and when $r = \cdot 4789\, a$ it is 5·7417.

The proportion of the mass $M$ which is included in radius $a/x$ is $\frac{1}{3}\beta^2\, dy/dx = \frac{1}{3}\beta^2 a_1\, dy_1/a\, dx_1 = \frac{\cdot 39723}{\cdot 6264}\, dy_1/dx_1$. Hence, the masses may be computed.

At any part of the isothermal sphere gravity $g$ is to be found from

$$g = \tfrac{1}{3}\beta^2 \mu M \frac{dy}{dx} \cdot \frac{x^2}{a^2};$$

or, expressing $g$ in terms of $G$ gravity at the surface, we have, since $G = \mu M/a^2$,

$$\frac{g}{G} = \tfrac{1}{3}\beta^2 x^2 \frac{dy}{dx}. \qquad (27)$$

The angular velocity of a body moving in a circular orbit at any part of the nebula, and its linear velocity $v$ are also easily to be found.

## § 5. *On an Atmosphere in Convective Equilibrium.*

I shall now suppose that a sphere of gas of mass $M$ at minimum temperature is bounded by an atmosphere in convective equilibrium, with continuity of temperature and density at the sphere of discontinuity of radius $a$. Let $v_0^2$ be the mean square of velocity of agitation in the isothermal sphere, and $v^2$ that at any other radius $r$. Then throughout the isothermal sphere $v^2 = v_0^2$, but in the layer outside $v^2$ gradually decreases to zero.

Let $w_0$ be the density and $p_0$ the pressure at radius $a$, and $w$, $p$ the same things at radius $r$.

Then, if the ratio of the two specific heats be that deduced from the simple kinetic theory of gases, without any allowance for intra-molecular vibrations, we have that ratio equal to $\frac{5}{3}$.

Hence,

$$p = p_0 \left(\frac{w}{w_0}\right)^{\frac{5}{3}},$$

and

$$\tfrac{1}{3} v^2 = p_0 \frac{w^{\frac{2}{3}}}{w_0^{\frac{5}{3}}} = \tfrac{1}{3} v_0^2 \cdot \left(\frac{w}{w_0}\right)^{\frac{2}{3}},$$

also

$$\frac{dp}{w} = \tfrac{5}{3} \cdot \frac{p_0}{w_0^{\frac{5}{3}}} w^{-\frac{1}{3}}\, dw = \tfrac{5}{2} \cdot \tfrac{1}{3} v_0^2\, d\left(\frac{w}{w_0}\right)^{\frac{2}{3}}.$$

Now, the equation for the hydrostatic equilibrium of the layer is

$$\frac{r^2\, dp}{w\, dr} + \mu M + 4\pi\mu \int_a^r w r^2\, dr = 0. \qquad (28)$$

Let

$$x = \frac{a}{r}, \qquad z = \frac{v^2}{v_0^2} = \left(\frac{w}{w_0}\right)^{\frac{2}{3}},$$

and we have

$$\frac{r^2\, dp}{w\, dr} = -\tfrac{5}{2} \cdot \tfrac{1}{3}\, v_0^2\, a \frac{dz}{dx},$$

$$\mu M = \frac{v_0^2\, a}{\beta^2},$$

$$4\pi\mu a^3 = \frac{3\mu M}{\rho} = \frac{\mu M}{w_0}, \text{ since } w_0 = \tfrac{1}{3}\rho \text{ rigorously,}$$

$$= \frac{v_0^2 a}{\beta^2 w_0}.$$

Hence, our equation is

$$\mu M \left\{ -\tfrac{5}{6}\beta^2 \frac{dz}{dx} + 1 + \int_x^1 \frac{z^{\frac{3}{2}}}{x^4}\, dx \right\} = 0. \quad \ldots \ldots \quad (29)$$

It is obvious the $\frac{5}{6}\beta^2 M\, dz/dx$ is the whole mass (expressed in terms of the mass of the isothermal sphere) enclosed inside of radius $a/x$. The differential equation to be satisfied is

$$\tfrac{5}{6}\beta^2 \frac{d^2z}{dx^2} + \frac{z^{\frac{3}{2}}}{x^4} = 0. \quad \ldots \ldots \ldots \ldots \quad (30)$$

We have seen in the last section that $\frac{1}{3}\beta^2 = \cdot 39723$, and, hence, $\frac{5}{6}\beta^2 = \cdot 99308$.

This equation is not so easy to solve as that in the last section, and I have not succeeded in finding the general law of the coefficients in an expansion. Nevertheless it is easy to find a series which will do all that is required.

Let $c$ be any value of $x$ for which we know $z$ and $dz/dx$, and let

$$\xi = x - c.$$

Assume

$$z = z_0 \{1 + A_1\xi + A_2\xi^2 + A_3\xi^3 + \ldots\}.$$

Then, if the suffix 0 indicates the value of a symbol when $x = c$ and $\xi = 0$, we have

$$z_0 = z_0,$$

$$\left(\frac{dz}{dx}\right)_0 = A_1 z_0,$$

$$\left(\frac{d^2z}{dx^2}\right)_0 = 2A_2 z_0.$$

But

$$\left(\frac{d^2z}{dx^2}\right)_0 = -\frac{6}{5\beta^2} \cdot \frac{z_0^{\frac{3}{2}}}{c^4},$$

and

$$2A_2 z_0 = -\frac{6}{5\beta^2}\cdot\frac{z_0^{\frac{3}{2}}}{c^4}, \qquad \text{or} \qquad A_2 = -\frac{3}{5\beta^2}\frac{z_0^{\frac{1}{2}}}{c^4},$$

so that, if $z_0$ and $(dz/dx)_0$ are known, $A_2$ is known.

The differential equation (30) which we have to satisfy is

$$\tfrac{5}{6}\beta^2(\xi + c)^4 \frac{d^2 z}{d\xi^2} = -z^{\frac{3}{2}},$$

or

$$\frac{1}{A_2}\left(\frac{\xi}{c} + 1\right)^4 \frac{d^2(z/z_0)}{d\xi^2} = 2\left(\frac{z}{z_0}\right)^{\frac{3}{2}}.$$

Now, by expansion,

$$\begin{aligned}
2\left(\frac{z}{z_0}\right)^{\frac{3}{2}} = 2 &+ 3A_1\xi + 3\left[A_2 + \tfrac{1}{4}A_1^2\right]\xi^2 + 3\left[A_3 + \tfrac{1}{2}A_1A_2 - \tfrac{1}{24}A_1^3\right]\xi^3 \\
&+ 3\left[A_4 + \tfrac{1}{2}(A_1A_3 + \tfrac{1}{2}A_2^2) - \tfrac{1}{8}A_1^2A_2 + \tfrac{1}{64}A_1^4\right]\xi^4 \\
&+ 3\left[A_5 + \tfrac{1}{2}(A_1A_4 + A_2A_3) - \tfrac{1}{8}(A_1^2A_3 + A_1A_2^2) + \tfrac{1}{16}A_1^3A_2 - \tfrac{1}{128}A_1^5\right]\xi^5 \\
&+ 3\left[A_6 + \tfrac{1}{2}(A_1A_5 + A_2A_4 + \tfrac{1}{2}A_3^2) - \tfrac{1}{8}(A_1^2A_4 + 2A_1A_2A_3 + \tfrac{1}{3}A_2^3)\right. \\
&\qquad \left. + \tfrac{1}{16}(A_1^3A_3 + \tfrac{3}{2}A_1^2A_2^2) - \tfrac{5}{128}A_1^4A_2 + \tfrac{7}{1536}A_1^6\right]\xi^6 \\
&+ 3\left[A_7 + \tfrac{1}{2}(A_1A_6 + A_2A_5 + A_3A_4) - \tfrac{1}{8}(A_1^2A_5 + 2A_1A_2A_4 + A_1A_3^2 + A_2^2A_3)\right. \\
&\qquad + \tfrac{1}{16}(A_1^3A_4 + 3A_1^2A_2A_3 + A_1A_2^3) - \tfrac{5}{128}(A_1^4A_3 + 2A_1^3A_2^2) \\
&\qquad \left. + \tfrac{1}{256}A_1^5A_2 - \tfrac{3}{1024}A_1^7\right]\xi^7 + \dots \qquad (31)
\end{aligned}$$

And

$$\begin{aligned}
\frac{1}{A_2}\left(\frac{\xi}{c}+1\right)^4 \frac{d^2}{d\xi^2}(z/z_0) = 2 &+ \left(3\,.\,2\frac{A_3}{A_2} + \frac{4}{c}\,.\,2\,.\,1\right)\xi + \left(4\,.\,3\,.\frac{A_4}{A_2} + \frac{4}{c}\,.\,3\,.\,2\frac{A_3}{A_2} + \frac{6}{c^2}\,.\,2\,.\,1\right)\xi^2 \\
&+ \left(5\,.\,4\frac{A_5}{A_2} + \frac{4}{c}\,.\,4\,.\,3\frac{A_4}{A_2} + \frac{6}{c^2}\,.\,3\,.\,2\frac{A_3}{A_2} + \frac{4}{c^3}\,.\,2\,.\,1\right)\xi^3 \\
&+ \left(6\,.\,5\frac{A_6}{A_2} + \frac{4}{c}\,.\,5\,.\,4\frac{A_5}{A_2} + \frac{6}{c^2}\,.\,4\,.\,3\frac{A_4}{A_2} + \frac{4}{c^3}\,.\,3\,.\,2\frac{A_3}{A_2} + \frac{1}{c^4}\,.\,2\,.\,1\right)\xi^4 \\
&+ \left(7\,.\,6\frac{A_7}{A_2} + \frac{4}{c}\,.\,6\,.\,5\frac{A_6}{A_2} + \frac{6}{c^2}\,.\,5\,.\,4\frac{A_5}{A_2} + \frac{4}{c^3}\,.\,4\,.\,3\frac{A_4}{A_2} + \frac{1}{c^4}\,.\,3\,.\,2\frac{A_3}{A_2}\right)\xi^5 \\
&+ \dots \qquad (32)
\end{aligned}$$

By equating the coefficients in (31) and (32) we are able to determine the $A$'s. The law of the series (32) is obvious, and sufficient of the series (31) is written down to enable us to find $A_9$. We can, however, obtain a good approximation to higher coefficients, because the coefficients in (31) become relatively unimportant.

We now begin the solution with

$$c = 1, \qquad z_0 = 1, \qquad \left(\frac{dz}{dx}\right)_0 = \frac{1}{\frac{5}{6}\beta^2} = 1\cdot0070, \qquad \left(\frac{d^2z}{dx^2}\right)_0 = -\frac{1}{\frac{5}{6}\beta^2} = -1\cdot0070.$$

Hence,

$$A_1 = 1\cdot0070, \qquad A_2 = -\cdot5035,$$

whence I compute

$$A_3 = +\cdot41782, \quad A_4 = -\cdot30068, \quad A_5 = +\cdot16175, \quad A_6 = -\cdot01306, \quad A_7 = -\cdot1333,$$
$$A_8 = +\cdot266, \quad A_9 = -\cdot378, \quad A_{10} = +\cdot48, \quad A_{11} = -\cdot6.$$

With these coefficients I find

$$\left.\begin{array}{llllll} \frac{r}{a} = \frac{12}{11}, & \frac{12}{10}, & \frac{12}{9}, & \frac{12}{8}, & \frac{12}{7}, & \frac{12}{6} \\ z = \cdot9123 & \cdot8160 & \cdot7089 & \cdot5887 & \cdot4525 & \cdot2982 \end{array}\right\} \quad . \quad . \quad (33)$$

Then, evaluating $x^{-4}z^{\frac{3}{2}}$, and combining the several values by the rules for integration of the calculus of finite differences, I find

$$\left.\begin{array}{llllll} \frac{r}{a} = \frac{12}{11}, & \frac{12}{10}, & \frac{12}{9}, & \frac{12}{8}, & \frac{12}{7}, & \frac{12}{6} \\ \frac{5}{6}\beta^2\frac{dz}{dx} = \ldots & 1\cdot21 & 1\cdot35 & 1\cdot527 & 1\cdot729 & 1\cdot9513 \end{array}\right\} \quad . \quad . \quad (34)$$

When $r = 2$, we begin a new series with

$$c = \tfrac{1}{2}, \qquad z_0 = \cdot2982, \qquad A_1 = \left(\frac{1}{z}\frac{dz}{dx}\right)_0 = \frac{1\cdot9513}{\cdot9907 \times \cdot2982} = +6\cdot5894,$$
$$A_2 = \left(\frac{1}{2z}\frac{d^2z}{dx^2}\right)_0 = \frac{-z_0^{\frac{1}{2}}}{2(\frac{1}{2})^4} = -4\cdot3686.$$

From these I compute $A_3 = -2\cdot744$, $A_4 = +21\cdot365$, $A_5 = -45\cdot409$, $A_6 = +9\cdot932$, $A_7 = +\cdot319$.

It appears that $z$ vanishes when $x - c = -\cdot141$ or $x = \cdot359$.

It follows, therefore, that four equidistant values of $x$ lying between $r = 2a$ and $r = a/\cdot359 = 2\cdot786\,a$ correspond to $x - c = 0$, $x - c = -\cdot047$, $x - c = -\cdot094$, $x - c = -\cdot141$.

For the first of these, where $r = 2\,a$, we have $z = \cdot2982$, and for the last, where $r = 2\cdot786\,a$, $z = 0$; and, when $x - c = -\cdot047$, or $r = a/\cdot453 = 2\cdot208\,a$, I find $z = \cdot2031$; and, when $x - c = -\cdot094$, or $r = a/\cdot406 = 2\cdot463\,a$, I find $z = \cdot1033$.

Finding $x^{-4}z^{\frac{5}{3}}$ for these four values and combining them by the rules of integration, I find

$$\tfrac{5}{6}\beta^2 \frac{dz}{dx} = 2{\cdot}1767, \quad \text{when } r = 2{\cdot}786\, a. \qquad \ldots \ldots \quad (35)$$

We thus see that the mass of the whole system is 2·1767 times the mass of the isothermal nucleus, and its radius is 2·786 times the radius of the nucleus.

The mass of the isothermal nucleus is thus 46 per cent. of the whole. M. RITTER, taking the ratio of the specific heats as $\frac{7}{5}$ instead of $\frac{5}{3}$, says that the proportion is about 40 per cent.

## § 6. *On a Gaseous Sphere in "Isothermal-Adiabatic" Equilibrium.*

M. RITTER calls a sphere, with isothermal nucleus and a layer in convective equilibrium above it, a case of isothermal-adiabatic equilibrium. Since the height of an atmosphere in convective equilibrium depends only on the temperature at the base, and since the isothermal nucleus in our numerical example is at minimum temperature, the thickness of the adiabatic layer is a minimum, and the isothermal nucleus a maximum.

We are now in a position to collect together all the numerical results of the last two sections in a form appropriate for our subsequent investigation. It will be convenient to refer all the densities and masses to the mean density and mass of the isothermal nucleus. Gravity may also be referred to gravity $G$ at the limit of the isothermal nucleus, and velocity to $v_0^2$, the mean square of velocity of agitation in the isothermal nucleus.



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TABLE III.—Isothermal-Adiabatic Sphere.

| | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Radius $\frac{r}{a}$ = | 0 | ·1596 | ·3193 | ·4789 | ·6385 | ·7982 | ·9578 | 1 | 1·0909 | 1·2 | 1·3333 | 1·5 | 1·7143 | 2·0 | 2·208 | 2·463 | 2·786 |
| Square of velocity of agitation $\frac{v^2}{v_0^2}$ = | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | ·912 | ·816 | ·709 | ·589 | ·452 | ·298 | ·203 | ·103 | 0 |
| Density $\frac{w}{\frac{1}{3}\rho}$ = | 32·14 | 23·52 | 11·86 | 5·742 | 3·024 | 1·759 | 1·115 | 1 | ·871 | ·737 | ·597 | ·452 | ·304 | ·163 | ·092 | ·033 | 0 |
| Mass in terms of $M$ = | 0 | ·0361 | ·1881 | ·3942 | ·5988 | ·7866 | ·9574 | 1 | .. | 1·207 | 1·349 | 1·527 | 1·729 | 1·951 | .. | .. | 2·177 |
| Gravity $\frac{g}{G}$ = | 0 | 1·4156 | 1·8467 | 1·7188 | 1·4685 | 1·2346 | 1·0436 | 1 | .. | ·8385 | ·7589 | ·6785 | ·5882 | ·4878 | .. | .. | ·28 |
| Square of velocity of satellite $\frac{v^2}{v_0^2}$ = | 0 | ·1896 | ·4945 | ·6908 | ·7869 | ·8269 | ·8388 | ·8391 | .. | ·844 | ·849 | ·854 | ·846 | ·818 | .. | .. | ·66 |
| $\log \frac{\frac{1}{3}\rho}{w}$ = | 8·4929 | 8·6286 | 8·9260 | 9·2410 | 9·5194 | 9·7547 | 9·9529 | ·0000 | ·0598 | ·1325 | ·2241 | ·3452 | ·5166 | ·7882 | 1·0385 | 1·4791 | ∞ |
| $\log \frac{1}{[w/\frac{1}{3}\rho]\,[v/v_0]}$ = | 8·4929 | 8·6286 | 8·9260 | 9·2410 | 9·5194 | 9·7547 | 9·9529 | ·0000 | ·0797 | ·1766 | ·2988 | ·4603 | ·6889 | 1·0510 | 1·3846 | 1·9722 | ∞ |
| $\log F_1 = \log \left[\frac{3\pi}{8\beta^2} \frac{g/G}{[v^2/v_0^2]\,[w/\frac{1}{3}\rho]}\right]$ = | − ∞ | 8·7745 | 9·1872 | 9·4712 | 9·6813 | 9·8413 | 9·9664 | 9·9950 | .. | ·1393 | ·2487 | ·4019 | ·6256 | ·9970 | .. | .. | ∞ |
| $\log F_2 = \log \left[\frac{1}{\pi} \frac{[g/G]^{\frac{1}{2}}\, x^{\frac{1}{2}}}{[v/v_0]\,[w/\frac{1}{3}\rho]}\right]$ = | − ∞ | 8·6053 | 8·8098 | 9·0213 | 9·2031 | 9·3523 | 9·4744 | 9·5029 | .. | 9·5967 | 9·6793 | 9·7909 | 9·9594 | ·2475 | .. | .. | ∞ |

## § 7. *On the Kinetic Energy of Agitation and its Distribution in an Isothermal-Adiabatic Sphere of Gas.*

We shall now consider what would be the distribution of kinetic energy in the nebula if each meteorite (or molecule) were to fall from infinity to the neighbourhood where we find it, and were to retain that energy afterwards. This will give the distribution of energy in a swarm of the supposed arrangement of density, if the rate of diffusion of kinetic energy were to be infinitely slow, and if there were no loss of energy through imperfect elasticity.

The square of the velocity of a satellite in a circular orbit is one half of the square of the velocity acquired by the fall from infinity to the distance of the satellite from the centre. If the concentration has proceeded as far as radius $r$, and if a meteorite falls from infinity to distance $r$, then, if $U$ be its velocity, and $v$ the velocity in a circular orbit at distance $r$,

$$\tfrac{1}{2} U^2 = v^2 = \tfrac{1}{3}\beta^2 \frac{\mu M}{a} \cdot x\frac{dy}{dx} = \tfrac{1}{3} v_0{}^2 x \frac{dy}{dx}, \text{ in the isothermal sphere,}$$

$$= \tfrac{5}{6}\beta^2 \frac{\mu M}{a} \cdot x\frac{dz}{dx} = \tfrac{5}{6} v_0{}^2 x \frac{dz}{dx}, \text{ in the adiabatic layer.}$$

In these formulæ, by the definitions of $y$ and $z$,

$$y = \log_e\left(\frac{9w}{\beta^2\rho}\right) \text{ in the first, and } z = \left(\frac{w}{w_0}\right)^{\frac{2}{3}} \text{ in the second.}$$

From these formulæ $v^2$ was computed in Table III. The value of $v^2$ or $\tfrac{1}{2}U^2$ gives what may be called the theoretical value of the kinetic energy, because it gives us a measure of the amount of redistribution of energy by diffusion and loss of energy by imperfect elasticity, which must take place before the whole system can assume the form of an isothermal adiabatic sphere.

We will now go on to consider the total potential energy lost in condensation.

We have seen that the potential energy lost by the fall of a single meteorite is $\tfrac{1}{3}v_0{}^2 x\, dy/dx$ in the isothermal part, and $\tfrac{5}{6}v_0{}^2 x\, dz/dx$ in the outer part.

Now, in the isothermal part a spherical element of mass is

$$-\tfrac{1}{3}M\beta^2 \cdot \frac{d^2y}{dx^2}\, dx,$$

and the energy lost by its fall is

$$-\tfrac{1}{9}M\beta^2 v_0{}^2 \cdot x\frac{dy}{dx}\frac{d^2y}{dx^2}\, dx.$$

Hence, the whole energy lost in the concentration of the isothermal nucleus is

$$\tfrac{1}{9}M\beta^2 v_0^{\,2} \cdot \int_1^\infty x \frac{dy}{dx}\frac{d^2y}{dx^2}\,dx.$$

But

$$-\int_1^\infty x \frac{dy}{dx}\frac{d^2y}{dx^2}\,dx = \int_1^\infty \frac{e^y}{x^3}\frac{dy}{dx}\,dx = 3\int_1^\infty \frac{e^y}{x^4}\,dx - e^{y_0}$$

$$= 3\left(\frac{dy}{dx}\right)_0 - e^{y_0}$$

$$= \frac{9}{\beta^2} - \frac{9w_0}{\beta^2\rho}.$$

Hence, the energy lost is $Mv_0^{\,2}\left(1 - \frac{w_0}{\rho}\right)$. But in an isothermal sphere of minimum temperature $w_0 = \frac{1}{3}\rho$, and thus the total lost energy is $\frac{2}{3}Mv_0^{\,2}$.

Again, in the adiabatic layer an element of mass is

$$-\tfrac{5}{6}M\beta^2 \frac{d^2z}{dx^2}\,dx = +\, M\frac{z^{\frac{3}{2}}}{x^4}\,dx,$$

and, therefore, the energy lost by its fall from infinity is

$$\tfrac{5}{6}Mv_0^{\,2} \cdot \frac{z^{\frac{3}{2}}}{x^3}\frac{dz}{dx}\,dx,$$

and the whole loss of energy is the integral of this from $x = 1$ to $x = \cdot 359$. When $x = 1$, $z = 1$, and when $x = \cdot 359$, $z = 0$. Hence

$$\int_{\cdot 359}^1 \frac{z^{\frac{3}{2}}}{x^3}\frac{dz}{dx}\,dx = \tfrac{2}{5} + \tfrac{6}{5}\int_{\cdot 359}^1 \frac{z^{\frac{5}{2}}}{x^4}\,dx.$$

Thus, the whole energy lost in the adiabatic layer is

$$Mv_0^{\,2}\left[\tfrac{1}{3} + \int_{\cdot 359}^1 \frac{z^{\frac{5}{2}}}{x^4}\,dx\right].$$

Add this to the energy found before for the isothermal part, and the whole lost energy of the system is found to be

$$Mv_0^{\,2}\left[1 + \int_{\cdot 359}^1 \frac{z^{\frac{5}{2}}}{x^4}\,dx\right]. \qquad (36)$$

Now let us evaluate the total kinetic energy existing in the form of agitation of molecules.

In the isothermal part it is clearly $\frac{1}{2}Mv_0^2$. In the adiabatic part it is half the element of mass into the square of velocity of agitation integrated through the layer, that is to say, $\frac{1}{2} . M \frac{z^{\frac{3}{2}}}{x^4} dx \times v^2$, and, since $z = \frac{v^2}{v_0^2}$, we have

$$\tfrac{1}{2}Mv_0^2 \left[ 1 + \int_{\cdot 359}^{1} \frac{z^{\frac{5}{2}}}{x^4} dx \right]$$

for the total internal kinetic energy of agitation. This is rigorously one-half of the energy lost in concentration.

Hence, if a meteor swarm concentrates into this arrangement of density, one half of the original energy is occupied in vaporising and heating parts of the meteorites on impact, and the other half is retained as kinetic energy of agitation.

I find by quadrature that $\int_{\cdot 359}^{1} \frac{z^{\frac{5}{2}}}{x^4} dx = \cdot 643$. Hence, the potential energy lost in concentration is $Mv_0^2 (1\cdot 643)$, and that part of it which is retained as energy of agitation is $\frac{1}{2}Mv_0^2 (1\cdot 643)$. The whole mass of the system is $2\cdot 1767\ M$, and we may, therefore, write these

$$\cdot 7548\ (2\cdot 1767\ M)\, v_0^2 \quad \text{and} \quad \tfrac{1}{2} \times \cdot 7548\ (2\cdot 1767\ M)\, v_0^2.$$

It is clear then that the average mean square of velocity of agitation of the *whole* system is $\cdot 7548\ v_0^2$.* Or, shortly, the average temperature is very nearly $\frac{3}{4}$ of the temperature of the isothermal nucleus.

It follows from this whole investigation that for any given mass of matter, arranged in an isothermal-adiabatic sphere of given dimensions, the actual velocities of agitation are determinable throughout.

## § 8. *On the "Sphere of Action."*

When two meteorites pass near to one another, each will be deflected from its straight path by the attraction of the other. The question arises as to whether the amount of such deflection can be so great that the passage of two meteorites near to one another ought to be estimated as an encounter in the kinetic theory.

We shall now, therefore, find the deflection of two meteorites, moving with the mean relative velocity, when they pass so close as just to graze one another.

The mean square of relative velocity in the isothermal portion is $2v_0^2$, and this may be taken as the square of the velocity at infinity in the relative hyperbola described The angle between the asymptotes of the hyperbola is the deflection due to this sort of encounter.

Let $\alpha$, $\epsilon$ be the semi-axis and eccentricity of the hyperbola. Then, if $\epsilon$ be large, the

* M. RITTER gives ·741 in place of ·755, but, as already remarked, he uses a different value for the ratio of the specific heats.

angle between the asymptotes is $1/\epsilon$; and, if $\frac{1}{2}s$ be the radius of either meteorite, the pericentral distance (when they graze) is $s$. Therefore,

$$s = \alpha(\epsilon - 1).$$

By the law of central orbits

$$2v_0^2 = \frac{\mu m}{\alpha}.$$

Therefore,

$$\epsilon = \frac{2v_0^2 s}{\mu m} + 1.$$

But, since $v_0^2 = \beta^2 \mu M/\alpha$, we have

$$\epsilon = 2\beta^2 \frac{Ms}{m\alpha} + 1.$$

The unity on the right-hand side is negligible, and, since $180/\pi\epsilon$ is the deflection in degrees, that deflection is

$$\frac{180}{2\pi\beta^2} \frac{m\alpha}{Ms} \text{ degrees.}$$

Now, if $\delta$ be the density of the body of a meteorite, $m = \frac{1}{6}\pi\delta s^3$, and, therefore, this expression becomes

$$\frac{15^\circ}{\beta^2} \times \frac{\delta\alpha s^2}{M}.$$

Let us find what $s$ must be if the deflection is $10^\circ$; we have

$$s = \beta\sqrt{\frac{2M}{3\alpha\delta}}.$$

We may, for a rough evaluation, take $\beta$ as unity instead of $\sqrt{6/5}$, and suppose $\alpha$ to be equal to the distance of Neptune from the Sun (viz., $4{\cdot}5 \times 10^{14}$ cm.), and, as a very high estimate of the value of $\delta$, let us suppose the density of a meteorite is 10. Then, since the Sun, $M_0 = 2 \times 10^{33}$ grammes, and $M$ is about a half of the Sun's mass, we have

$$s = \left[\frac{2 \times 10^{33}}{3 \times 4{\cdot}5 \times 10^{14} \times 10}\right]^{\frac{1}{2}} = (15 \times 10^{16})^{\frac{1}{2}} = 4 \times 10^8.$$

Hence, $m = \frac{1}{6}\pi\delta s^3 = \frac{1}{6}\pi \times 10 \times 64 \times 10^{24} = 3 \times 10^{26}$ grammes, in round numbers.

But the Earth's mass is $6 \times 10^{27}$ grammes, and therefore the meteorites are one-twentieth of the mass of the Earth.

It follows, therefore, that, with such small masses as those with which the present theory deals, the deflection due to gravity is insensible, and we need only estimate actual impacts as encounters.

Hence, the radius of the sphere of action of a meteorite is identical with the diameter of its body.

§ 9. *On the Criterion for the Applicability of Hydrodynamics to a Swarm of Meteorites.*

The question at issue is to determine within what limits the quasi-gas formed by a swarm of colliding meteorites may be treated as a plenum, subject to the laws of hydrodynamics. The doctrines of the nebular hypothesis depend on the stability of a rotating mass of fluid, and that stability depends on the frequencies of its gravitational oscillations. Now the works of POINCARÉ and others seem to show that instability, at least in a homogeneous fluid, first arises from one of the graver modes of oscillation, and the period of the gravest mode does not differ much from the period of a satellite grazing the surface of the mass of fluid. Then, in order that hydrodynamical treatment should be applicable for the discussion of such questions of stability, the mean free time between collisions must be small compared with the period of such a satellite. Another way of stating this is that the mean free path of a meteorite shall be but little curved, and that the velocity of a meteorite shall be but little changed by gravity in the interval between two collisions. This must be fulfilled not only at the limits of the swarm, but at every point of it. The condition above stated will be satisfied if the space through which a meteorite falls from rest, at any part of the swarm, in the mean interval between collisions is small compared with the mean free path. If this criterion is fulfilled, then, in most respects which we are likely to discuss, the swarm will behave like a gas, and we must at present confine the consideration of the matter to this general criterion.

It would be laborious to determine exactly the space fallen through from rest, because gravity varies as the meteorite falls, but a sufficiently close approximation may be found by taking gravity constant throughout the fall and equal to its value at the point from which the meteorite starts.

We have already denoted by $g$ the value of gravity at any part of the swarm, and have tabulated it in Table III. in terms of $G$ or $\mu M/a^2$.

Now the mean interval is $T = L/(v\sqrt{8/3\pi})$. Hence, if $D$ be the distance fallen in this time,

$$D = \tfrac{1}{2}\, gT^2 = \tfrac{1}{2}\frac{gL^2}{v^2}\cdot\frac{3\pi}{8}.$$

But

$$L = l\left(\frac{M_0}{M}\right)\frac{\frac{1}{3}\rho}{w}\cdot\frac{m}{s^2} \quad \text{and} \quad \frac{G}{{v_0}^2} = \frac{1}{\beta^2 a}.$$

Therefore,

$$\frac{D}{L} = \frac{l}{2a}\cdot\frac{M_0}{M}\cdot\frac{m}{s^2}\cdot\left\{\frac{3\pi}{8\beta^2}\cdot\frac{[g/G]}{[v^2/{v_0}^2][w/\frac{1}{3}\rho]}\right\} = \frac{l}{2a}\cdot\frac{M_0}{M}\cdot\frac{m}{s^2}\cdot F_1. \quad . \quad . \quad . \quad (37)$$

The factor $F_1$ has been tabulated above in Table III., and it increases from the centre to the outside.

This criterion may be regarded from another point of view, for, if the meteorite be describing a circular orbit about the centre of the swarm, $D$ is the deflection from the straight path in the mean interval between two collisions. Then the criterion is that the deflection shall be small compared with the mean free path.

We may consider the criterion from again another point of view, and state that the arc of circular orbit described in the mean interval shall be a small fraction of the whole circumference.

The linear velocity $v$ in the circular orbit is given by

$$v^2 = g\frac{a}{x} = \frac{v_0^2}{\beta^2 x} \cdot \frac{g}{G}.$$

And the mean interval $T = L/[v\sqrt{8/3\pi}]$. Hence, if $A$ be the arc described with velocity $v$ in time $T$,

$$A^2 = \frac{3\pi}{8\beta^2} \cdot \frac{L^2}{v^2/v_0^2} \cdot \frac{g}{Gx} = \frac{L^2}{v^2/v_0^2} \cdot \frac{g}{Gx} \text{ nearly, since } \frac{3\pi}{8\beta^2} = \cdot 988.$$

But the whole arc of circumference $C$ is $2\pi a/x$.

Therefore,

$$\frac{A}{C} = \frac{L}{2a} \cdot \frac{1}{\pi} \cdot \frac{[g/G]^{\frac{1}{2}} x^{\frac{1}{2}}}{v/v_0}$$

$$= \frac{l}{2a} \cdot \frac{M_0}{M} \cdot \frac{m}{s^2} \cdot \left\{\frac{1}{\pi} \cdot \frac{[g/G]^{\frac{1}{2}} x^{\frac{1}{2}}}{[v/v_0][w/\frac{1}{3}\rho]}\right\} = \frac{l}{2a} \cdot \frac{M_0}{M} \cdot \frac{m}{s^2} \cdot F_2. \quad . \quad . \quad . \quad (38)$$

The factor $F_2$ has been tabulated above, in Table III.

## § 10. *On the Density of Meteorites and Numerical Application.*

It is necessary to make assumptions both as to the mass and the density of the meteorites. We have a right to assume, I think, that the density $\delta$ is a little less than that of iron, say about 6, and we may put $\frac{4}{3}\pi\delta$ equal to 25. Then we have

$$m = \tfrac{1}{6}\pi\delta s^3 = \tfrac{25}{8}s^3, \quad \text{and} \quad \frac{m}{s^2} = \tfrac{25}{8}s.$$

There is but little information about the average size of meteorites; but, if we retain the symbol $s$, it will be easy, by merely shifting the decimal point in the final results, to obtain results for all sizes. Thus, if $s = 1$ cm., $m = 3\frac{1}{8}$ grammes; if $s = 10$ cm., $m = 3\frac{1}{8}$ kilogrammes; if $s = 100$ cm., $m = 3\frac{1}{8}$ tonnes, and if $s = 1000$ cm., $m = 3125$ tonnes. I shall, therefore, keep $s$ in the analytical formulæ, and put it equal to unity in the numerical results.

In the first place, making no assumptions as to the density or masses of the meteorites, we have

$$M_0 = 2 \cdot 1767 \times M, \quad \tfrac{1}{3}\beta^2 = \cdot 39723.$$

Then, by substitution in (10) and (11), we have

$$\left.\begin{aligned} v_0 &= u \times 10^{9\cdot86918-10} \\ L &= l \times 10^{0\cdot33781}\frac{m/s^2}{w/\frac{1}{3}\rho} \\ T &= \tau \times 10^{0\cdot46863}\frac{m/s^2}{[v/v_0]\,[w/\frac{1}{3}\rho]} \\ \frac{D}{L} &= \frac{l}{2a} \times 10^{0\cdot33781} \times F_1 \times \frac{m}{s^2} \\ \frac{A}{C} &= \frac{l}{2a} \times 10^{0\cdot33781} \times F_2 \times \frac{m}{s^2} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (39)$$

We will now apply this solution to a case which will put the theory to a severe test. Suppose that the limit of the sphere of uniformly distributed energy of agitation is nearly as far as the planet Uranus, so that, say $a = 16a_0$. Then the extreme limit of the swarm is at $44\frac{1}{2}a_0$; but the orbit of the planet Neptune is at $30a_0$, so that the limit is further beyond Neptune than Saturn is from the Sun.

Now, if $a/a_0 = 16$, I find

$$\left.\begin{aligned} u &= 10^{5\cdot86117} \text{ cm. per sec.} \\ &= 10^{\cdot18795}a_0 \text{ per annum,} \\ \tau &= 10^{4\cdot46808} \text{ seconds} \\ &= 10^{6\cdot96899-10} \text{ years} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (40)$$

Introducing these values in (39) and putting $\frac{25}{8}s$ for $m/s^2$, I find

$$\left.\begin{aligned} v &= 1\cdot141a_0 \text{ per annum} = 5\cdot374 \text{ kilom. per sec.} \\ \frac{L}{a_0} &= 10^{7\cdot9540-10} \times \frac{s}{[w/\frac{1}{3}\rho]} \\ T &= 10^{7\cdot9325-10} \times \frac{s}{[v/v_0]\,[w/\frac{1}{3}\rho]} \\ \frac{D}{L} &= 10^{6\cdot4489-10}sF_1 \\ \frac{A}{C} &= 10^{6\cdot4489-10}sF_2 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (41)$$

Now we have in Table III. the logarithms of the several factors, which occur last in these formulæ (41), at various distances from the centre.

It will suffice for our purpose only to take every other value from Table III. The distances from the centre are expressed in terms of the astronomical unit distance, viz., the Earth's mean distance from the Sun. The mean free path is expressed both in the same unit and in kilometres; and the mean intervals between collisions in days. The criteria $D/L$ and $A/C$ are, of course, pure numbers. Table IV., as it stands, is applicable to meteorites weighing $3\frac{1}{8}$ grammes, but by shifting the decimal point one place to the right in the last four rows of entries it becomes kilogrammes, one more and it becomes tonnes, and another, thousands of tonnes, and so on.

## IV.—Table of Results.

The meteorites weigh $3\frac{1}{8}$ grammes, and have the density of iron. The swarm extends to $44\frac{1}{2}a_0$, $a_0$ being Earth's distance from Sun.

| | Sun. | Asteroids. | Saturn. | | | Uranus. | | Neptune. | |
|---|---|---|---|---|---|---|---|---|---|
| Distance from centre $\frac{r}{a_0}=$ | 0 | 2·55 | 7·66 | 12·77 | 16 | 19·2 | 24 | 32 | $44\frac{1}{2}$ |
| Velocity of mean square in kilometres per sec. $v=$ | 5·37 | 5·37 | 5·37 | 5·37 | 5·37 | 4·85 | 4·12 | 2·93 | 0 |
| Mean free path, $\frac{L}{a_0}=$ | ·00028 | ·00038 | ·00157 | ·00511 | ·00900 | ·0122 | ·0199 | ·0552 | $\infty$ |
| $L$ kilom. $=$ | 41,600 | 57,000 | 233,000 | 760,000 | 1,340,000 | 1,810,000 | 2,960,000 | 8,210,000 | $\infty$ |
| Mean free time, in days $T=$ | ·097 | ·133 | ·545 | 1·78 | 3·13 | 4·70 | 9·02 | 35·17 | $\infty$ |
| Criterion, $\frac{D}{L}=$ | . . | ·0000167 | ·0000832 | ·000195 | ·000278 | ·000387 | ·000709 | ·00279 | $\infty$ |
| Criterion, $\frac{A}{C}=$ | . . | ·0000113 | ·0000295 | ·0000633 | ·0000895 | ·000111 | ·000174 | ·000497 | $\infty$ |

The incidence of the several planets in the scale of distance is roughly indicated by the names written above.

The criteria show that, if the meteorites weigh $3\frac{1}{8}$ kilogrammes, the collisions are frequent enough, even beyond the orbit of Neptune, to allow the kinetic theory of gases to be applicable for such problems as are in contemplation. For, when $r/a = 32$, the two criteria (with decimal point shifted one place to the right) are ·028 and ·005, both small fractions. But, if the meteorites weigh $3\frac{1}{8}$ tonnes, the criteria cease to be very small, about $r/a = 24$. If they weigh 3125 tonnes, the applicability will cease somewhat beyond where the asteroids now are.

I conclude, then, from this discussion that we are justified in applying hydrodynamical treatment to a swarm of meteorites from which the solar system originated, even in the earliest stages of the history of the swarm.

This discussion has, of course, no bearing on the fundamental hypothesis that meteorites *can* glance from one another on impact with a virtually high degree of elasticity; nor does it do anything to justify the assumption that a swarm will consist approximately of a quasi-isothermal nucleus with a quasi-adiabatic layer over it. This latter assumption I have been led to by the considerations to which we now pass.

### § 11. *On the Diffusion of Kinetic Energy and on the Viscosity.*

In order to discuss these questions, it will be well to begin with a simple case of fluid motion.

Consider two-dimensional motion, in which there are a number of streams of equal breadth moving parallel to $y$ with velocity $V$, and, interpolated between them, let there be strata of quiescent fluid; suppose then that we wish to find the motion at any time after this initial state. Let the boundaries of the streams $V$ be from $x = ml$ to $\frac{1}{2}(2m + 1)l$. Then, if $u$ be the velocity at $x$, parallel to $y$ at time $t$, and $\nu$ the kinetic modulus of viscosity, the equation of motion is

$$\frac{du}{dt} = \nu \frac{d^2u}{dx^2}.$$

The solution of this being of the form $e^{-p^2t} \cos px$, the complete solution satisfying the initial condition is—

$$u = \tfrac{1}{2} V + \frac{2V}{\pi}\left[e^{-\pi^2\nu t/l^2} \cos \frac{\pi x}{l} - \tfrac{1}{3} e^{-9\pi^2\nu t/l^2} \cos \frac{3\pi x}{l} + \tfrac{1}{5} e^{-25\pi^2\nu t/l^2} \cos \frac{5\pi x}{l} - \ldots\right].$$

Now, if we refer time to a period $\tau$, where $\tau = l^2/\pi^2\nu$, then after a time $\theta\tau$, which is greater than $\tau$, the solution is sensibly

$$u = \tfrac{1}{2} V\left[1 + \frac{4}{\pi e^\theta} \cos \frac{\pi x}{l}\right].$$

It is clear that the maximum of $u$ occurs when $x = 0$, and the minimum when $x = l$, and that they are

$$\tfrac{1}{2} V\left[1 \pm \frac{4}{\pi e^\theta}\right].$$

Hence, the difference between the maximum and minimum is $4V/\pi e^\theta$. Therefore, the ratio of the greatest difference of velocities after time $\theta\tau$ to the initial difference of velocities is $4/\pi e^\theta$. When $\theta$ is 1, 2, 3, this ratio assumes the values 1/2·135, 1/5·804, 1/15·73 respectively. Thus, after three times the interval $\tau$, the difference of velocities is small. The time $\tau$ may be therefore taken as a convenient measure of viscosity.

In our problem the streams must be taken of a width comparable with the linear dimensions of the solar system. I therefore take $l$, the width of the streams, as equal to $a_0$, the Earth's distance from the Sun, and we have

$$\tau = \frac{a_0^2}{\pi^2\nu}.$$

Now, according to the kinetic theory of gases, the kinetic modulus of viscosity is $1/\pi$ into the mean free path multiplied by the mean velocity. Hence,

$$\nu = \frac{1}{\pi} L \left( v \sqrt{\frac{8}{3\pi}} \right).*$$

Hence, we have

$$\tau = \left(\frac{a_0}{L}\right)\left(\frac{a_0}{v}\right) \sqrt{\frac{3}{8\pi}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (42)$$

If we apply this formula to the solution which has been already found in Table IV., we obtain the following results :—

| $\frac{\ }{a_0}$ | 0, | 2·55, | 7·66, | 12·77, | 16, | 19·2, | 24, | 32, |
|---|---|---|---|---|---|---|---|---|
| $\tau$ years | 1082, | 792, | 193, | 59·2, | 33·7, | 27·5, | 19·8, | 61·7. |

These results are applicable to meteorites weighing $3\frac{1}{8}$ grammes in a swarm extending to $44\frac{1}{2}\ a_0$. If the meteorites weigh $3\frac{1}{8}$ kilogrammes, the values of $\tau$ would be one-tenth of the tabulated values. If the streams were ten times as broad, the periods would be a hundred times as long.

Now the periods $\tau$ in the above table, even if multiplied by a thousand, must be considered as short in the history of a stellar system. It thus appears that the quasi-viscosity must be such that a swarm of meteors must, if revolving, move nearly without relative motion of its parts, at least in the early stages of its evolution.

But let us consider the values of $\tau$ at different epochs in the history of the same system. If $a$ be the radius of the isothermal sphere the formulæ (9) and (10) show that $L/a_0$ varies as $a^3$, whilst $v/a_0$ varies as $a^{-\frac{1}{2}}$. Hence $\tau$ varies inversely as $a^{\frac{5}{2}}$. Thus, as the swarm contracts, the periods $\tau$ increase rapidly.

Thus, later in the history, the viscosity will probably fall off so much that equalisation of angular velocity may be no longer attained, and we should then have the central portion rotating more rapidly than the outside, with a gradual transition from one angular velocity to the other.

The modulus $\nu$ gives, besides the viscosity, the rate of equalisation of the kinetic energy of agitation ; this corresponds in a true gas with the conduction of heat. The conclusion at which we thus arrive appears to justify the assumption that the whole of the central part of the swarm is endued with uniform kinetic energy of agitation, and that the mass of the quasi-isothermal nucleus is the greatest possible. With regard to the assumption that the nucleus is coated with a layer in adiabatic or convective equilibrium, it may be remarked that the velocity of agitation must decrease when we get to the outskirts of the swarm, and convective equilibrium will probably satisfy the conditions of the case better than any other. Further considerations will be adduced on this point in the Summary.

* MEYER, 'Kinetische Theorie der Gase,' p. 321. The $1/\pi$ is derived from a numerical quadrature which gives the value ·318, and it is apparently only accidentally equal to $1/\pi$. The $v\sqrt{(8/3\pi)}$ is the mean velocity denoted $\Omega$ by MEYER.

## § 12. *On the Rate of Loss of Kinetic Energy through Imperfect Elasticity, and on the Heat Generated.*

In a collision between two meteorites the loss of energy is probably proportional to their relative kinetic energy before impact. Therefore, the amount of heat generated by a single meteorite per unit time is proportional to the kinetic energy (say $h$) and to the frequency of collision. By (10) the frequency, or reciprocal of T, varies as $vws^2/m$; but $m^{\frac{1}{2}}v$ is equal to $(2h)^{\frac{1}{2}}$, and $s^2m^{-\frac{3}{2}}$ varies as $m^{-\frac{5}{6}}$. Hence, the frequency of collision varies as $h^{\frac{1}{2}}wm^{-\frac{5}{6}}$, and the amount of heat generated by a single meteorite per unit time varies as $h^{\frac{3}{2}}wm^{-\frac{5}{6}}$. But, if $p$ be the quasi-hydrostatic pressure, $p$ varies as $hwm^{-1}$, and, therefore, the heat generated by a single meteorite varies as $h^{\frac{1}{2}}pm^{\frac{1}{6}}$.

Then, to find the total heat generated per unit time and volume, we have to multiply this by the number of meteorites per unit volume, that is to say, by $wm^{-1}$, which is equal to $3ph^{-1}$.

Thus the amount of heat generated per unit time and volume is proportional to $p^2m^{\frac{1}{6}}h^{-\frac{1}{2}}$. With meteorites of uniform size, and with uniform kinetic energy of agitation, this becomes simply the square of the hydrostatic pressure.

The mean temperature of the gases volatilised by collisions must depend on a variety of considerations, but it would seem as if the temperature would follow, more or less closely, the variations of heat generated per unit time and volume.

## § 13. *On the Fringe of a Swarm of Meteorites.*

The law of distribution of meteorites found above depends on the frequency of collisions. But at some distance from the centre collisions must have become so rare that the statistical method is inapplicable. There must then be a sort of fringe to the swarm, which I attempt to represent by supposing that beyond a certain radius $a$ (not the same as the former $a$) collisions never occur, and each meteorite describes an orbit under gravity.

Now, at any point gravity depends on the mass of all the matter lying inside a sphere whose radius is equal to the distance of that point from the centre of the swarm. Hence, the value of gravity depends on the law of density of distribution of the meteorites, which is the thing which we are seeking to discover.

We suppose, then, that from every point of a sphere of radius $a$ a fountain of meteorites is shot up, at all inclinations to the vertical, and with velocities grouped about a mean velocity, according to the exponential law appropriate to the case. As many meteorites are supposed to fall back on to the surface as leave it, and this inward cannonade against the boundary of the sphere exactly balances the quasi-gaseous pressure on the inside of the sphere. Thus, the ideal surface may be annihilated. Since the falling half of the orbit of a meteorite is the facsimile of the rising half, we need only trace the body from projection to apocentre, and then double the distribution of density which is deduced on the hypothesis that all the meteorites are rising. Again, since every element of the sphere shoots out a similar fountain, and since collisions are precluded by hypothesis, we need only consider the velocity along the radius vector. As far as concerns the distribution of density, it is the same as if each element shot up a vertical fountain; but, of course, in determining the vertical velocity, we must pay attention to the inclination to the vertical at which the meteorite was shot out.

The mass of the matter inside the sphere, whose attraction affords the principal part of the force under which the meteorites move, is say $M$, and, for the sake of simplicity of notation, we shall take $2\mu M/a$ as being unit square of velocity.

Now, let $\frac{1}{2}\phi(r)$ be the potential at the point whose radius is $r$, and suppose that a meteorite is shot out from a point on the sphere with a velocity $u$, and at an inclination $\epsilon$ to the vertical; then, if $r$, $\theta$ be the radius vector and longitude of the meteorite at the time $t$, the equations of conservation of moment of momentum, and of energy are—

$$r^2 \frac{d\theta}{dt} = ua \sin \epsilon,$$

$$\left(\frac{dr}{dt}\right)^2 + \left(r \frac{d\theta}{dt}\right)^2 - \phi(r) = u^2 - \phi(a).$$

If we write $f(r) = \phi(a) - \phi(r)$, and eliminate $d\theta/dt$, we get

$$r^2 \frac{dr}{dt} = r\left\{r^2\left(u^2 - f(r)\right) - u^2 a^2 \sin^2 \epsilon\right\}^{\frac{1}{2}}.$$

Now, we are to regard $dr/dt$ as the vertical velocity in a fountain squirting up from a point on the sphere. Then, since $f(a) = 0$, it follows that at the foot of the fountain $r^2\, dr/dt$ is equal to $a^2 u \cos \epsilon$. If, therefore, $\delta$ be the density at the height $r$, and $\delta_0$ at the foot, the equation of continuity is

$$\delta\, r^2 \frac{dr}{dt} = \delta_0\, a^2 u \cos \epsilon.$$

Therefore,

$$\frac{\delta}{\delta_0} = \frac{a^2 u \cos \epsilon}{r\left\{r^2\left(u^2 - f(r)\right) - u^2 a^2 \sin^2 \epsilon\right\}^{\frac{1}{2}}}.$$

But now let us suppose that the meteorites are not only shot out at inclination $\epsilon$, but at all possible inclinations from 0° and 90°. It is then clear that this expression must be multiplied by $\sin \epsilon\, d\epsilon$, and integrated. Hence, if $\delta$ now denotes the integral density,

$$\delta = C \int_0^{\epsilon_0} \frac{a^2 u \cos \epsilon \sin \epsilon\, d\epsilon}{r\left\{r^2\left(u^2 - f(r)\right) - u^2 a^2 \sin^2 \epsilon\right\}^{\frac{1}{2}}},$$

where $C$ is a constant which it will be unnecessary to determine, and where the limit $\epsilon_0$ will be the subject of future consideration.

Effecting the integration, we have

$$\delta = -\frac{C}{ur}\{r^2(u^2 - f(r)) - u^2a^2\sin^2\epsilon\}^{\frac{1}{2}}, \text{ between limits,}$$

$$= \frac{C}{ur}[\{r^2(u^2 - f(r))\}^{\frac{1}{2}} - \{r^2(u^2 - f(r)) - u^2a^2\sin^2\epsilon_0\}^{\frac{1}{2}}].$$

It is obvious that, if $u^2$ is greater than $r^2f(r)/(r^2 - a^2)$, the square root involved in $dr/dt$ does not vanish for any value of $\epsilon$; and, hence, we must simply take $\epsilon_0 = 90°$. If, on the other hand, $u^2$ is less than this critical value, $\epsilon_0$ is that value of $\epsilon$ which makes $dr/dt$ vanish.

Thus, our formula divides into three, viz. :—

1st. $u^2$ greater than $\dfrac{f(r)}{1 - a^2/r^2}$;

$$\delta = \frac{C}{u}\left[(u^2 - f(r))^{\frac{1}{2}} - \left(1 - \frac{a^2}{r^2}\right)^{\frac{1}{2}}\left(u^2 - \frac{f(r)}{1 - a^2/r^2}\right)^{\frac{1}{2}}\right].$$

2nd. $u^2$ less than $\dfrac{f(r)}{1 - a^2/r^2}$;

$$\delta = \frac{C}{u}(u^2 - f(r))^{\frac{1}{2}}.$$

3rd. $u^2$ less than $f(r)$; $\delta = 0$.

The physical meaning of this division is as follows: If we take a station near the surface of the sphere, meteorites shot out at all inclinations, even horizontally, reach the height of our station; and, when they are shot out horizontally, $\epsilon = 90°$. If we go, however, to a higher region, there is a certain inclination which just brings the meteorites at apocentre, where $dr/dt = 0$, to our height; but those shot out more nearly horizontally fail to reach us. Still higher, not even a meteorite shot up vertically can reach us, and the density vanishes.

These results only correspond to a single velocity $u$; but, if $v^2$ be the mean square of the velocity, the number of meteorites whose velocities range between $u$ and $u + du$ is proportional to $u^2e^{-3u^2/2v^2}du$.* Hence, we have to multiply $\delta$ by this expression, and integrate from $u = \infty$ to $u = 0$.

Now, the first term of the first form for $\delta$ is the same as the second form; and in the third form $\delta$ is zero; hence, this first term when multiplied by the exponential has to be integrated from $u^2 = \infty$ to $f(r)$. The second term of the first form of $\delta$ has to be multiplied by the exponential, and integrated from $u^2 = \infty$ to $f(r)/(1 - a^2/r^2)$.

Now, for the first term put

$$\frac{3}{2v^2}(u^2 - f(r)) = x^2;$$

* Oskar Meyer, 'Die Kinetische Theorie der Gase,' 1877, pp. 271–2.

therefore,

$$u\,(u^2 - f(r))^{\frac{1}{2}}\,du = \left(\frac{2v^2}{3}\right)^{\frac{3}{2}} x^2\,dx,$$

and the limits of $x$ are $\infty$ to 0.

Hence, the first term is

$$C\left(\frac{2v^2}{3}\right)^{\frac{3}{2}} e^{-3f(r)/2v^2}\int_0 x^2\,e^{-x^2}\,dx$$

Again, for the second term put

$$\frac{3}{2v^2}\left(u^2 - \frac{f(r)}{1 - a^2/r^2}\right) = x^2,$$

and similarly introduce it into the second term, and we have

$$-\,C\left(\frac{2v^2}{3}\right)^{\frac{3}{2}}\left(1 - \frac{a^2}{r^2}\right)^{\frac{1}{2}} e^{3f(r)/2v^2(1-a^2/r^2)}\int_0^\infty x^2\,e^{-x^2}\,dx.$$

From these expressions we may omit the constant factors; and, if $w$ be the density at height $r$, whilst $w_0$ is the density at the sphere,

$$\frac{w}{w_0} = e^{-3f(r)/2v^2} - \left(1 - \frac{a^2}{r^2}\right)^{\frac{1}{2}} e^{-3f(r)/2v^2(1-a^2/r^2)}.$$

In this formula unit square of velocity is $2\mu M/a$; but we have elsewhere written $v^2 = \beta^2\mu M/a$; hence, if the special unit of velocity be given up, we may write $\beta^2$ in place of $2v^2$, and the result becomes

$$\frac{w}{w_0} = e^{-3f(r)/\beta^2} - \left(1 - \frac{a^2}{r^2}\right)^{\frac{1}{2}} e^{-3f(r)/\beta^2(1-a^2/r^2)}. \qquad (43)$$

It is interesting to observe the connection between this law of density and that which would have held if the gaseous law (due to collisions) had obtained. In that case, since $\frac{1}{2}\phi(r)$ is the potential, we should have had

$$\frac{1}{w}\frac{dp}{dr} - \frac{1}{2}\frac{d}{dr}\phi(r) = 0.$$

Now, $p = \frac{1}{3}v^2 w$, and, therefore,

$$\log w - \frac{3}{2v^2}\phi(r) = \text{const.}$$
$$= \log w_0 - \frac{3}{2v^2}\phi(a).$$

Thus,

$$\log\frac{w}{w_0} = -\frac{3}{2v^2}[\phi(a) - \phi(r)] = -\frac{3}{2v^2}f(r),$$

or

$$\frac{w}{w_0} = e^{-3f(r)/2v^2} = e^{-3f(r)/\beta^2}.$$

The first term of our result, then, is exactly that resulting from the gaseous law, and the second subtractive term represents the action of the diminished velocity with which the meteorites move in the higher regions, when they are liberated from the equalising effects of continual impacts.

By previous definition, $\mu M f(r)/a$ is the excess of the potential at radius $a$ above its value at radius $r$; hence,

$$\frac{\mu M}{a} f(r) = \int_a^r \frac{4\pi\mu}{r^2} \int_0^r wr^2\, dr\,.\, dr.$$

Now, since $f(r)$ is only required for values of $r$ greater than $a$, we may put $w$ equal to its mean value $\rho$, between the limits 0 and $a$. Thus,

$$\int_0^r wr^2\, dr = \int_a^r wr^2\, dr + \tfrac{1}{3}\rho a^3.$$

Hence,

$$f(r) = \frac{4\pi a}{M}\left[\int_a^r \frac{1}{r^2}\int_a^r wr^2\, dr + \tfrac{1}{3}\rho a^3 \int_a^r \frac{dr}{r^2}\right] = \left(1 - \frac{a}{r}\right) + 3\int_1^{r/a} \frac{1}{z^2}\int_1^z \frac{w}{\rho} z^2\, dz.$$

If this form for $f(r)$ were substituted in (43), we should obtain a very complicated differential equation for $w$. We may, however, find two values of $f(r)$ within which the truth must lie.

*First*, if we neglect the attraction of all the matter lying outside of radius $a$, the second term vanishes, and we have,

$$f(r) = 1 - \frac{a}{r};$$

and the law of density is

$$\frac{w}{w_0} = e^{-3(1-a/r)\beta^2} - \left(1 - \frac{a^2}{r^2}\right)^{\frac{1}{2}} e^{3/(1+a/r)\beta^2}. \quad . \quad . \quad . \quad . \quad . \quad (44)$$

*Secondly*, we may suppose the density to go on diminishing according to the inverse square of the distance. We have seen in the preceding solution and tables that this is roughly the law of diminution for a long way outside the isothermal nucleus. According to this assumption, $w = w_0\, a^2/r^2$. Hence, in the second term of $f(r)$ we put $w = w_0\, a^2/r^2 = w_0/z^2$.

Hence,

$$\int_1^z \frac{w}{\rho} z^2\, dz = \frac{w_0}{\rho}\int_1^z dz = \frac{w_0}{\rho}(z - 1),$$

and

$$f(r) = 1 - \frac{a}{r} + \frac{w_0}{\frac{1}{3}\rho}\int_1^{r/a} \frac{z-1}{z^2}\, dz = \left(1 - \frac{w_0}{\frac{1}{3}\rho}\right)\left(1 - \frac{a}{r}\right) + \frac{w_0}{\frac{1}{3}\rho}\log\frac{r}{a}. \quad . \quad (45)$$

The substitution of this value in (43) gives the law of density.

In order to see the kind of results to which these formulæ lead, let us suppose that, when we have reached radius 2 in the adiabatic layer, collisions have become so rare as to be negligible. Then the symbols in the formulæ of this section have numerical values; and, in order to distinguish them, let them be accented, so that, for example, we write $\alpha'$, $\beta'^2$, $\rho'$, &c., in (43), (44), and (45), in place of $\alpha$, $\beta^2$, $\rho$.

Now Table III. shows that, when $a' = 2a$, $M' = 1\cdot95\,M = 2M$ nearly. Hence, $M'/a' = M/a$ nearly. But, at radius $2a$ in Table III., $v^2/v_0^2 = \cdot298 = \cdot3$, and this $v^2$ is what we now write $v'^2$ or $\beta'^2\mu M'/a'$, whilst $v_0^2 = \beta^2\mu M/a$.

But $\beta^2 = \frac{6}{5}$ very nearly; hence, $\beta'^2/\beta^2 = \cdot3$, or $\beta'^2 = \cdot36$.

Thus, $3/\beta'^2 = 8\cdot333$.

Then, substituting $2a$ for $a'$, and noticing that in Table III., $w/\frac{1}{3}\rho = \cdot163$, when $r = 2a$, the first law of density (44) becomes

$$\frac{w}{\frac{1}{3}\rho} = \cdot163\left[e^{-\frac{25}{3}(1-2a/r)} - \left(1 - 4\frac{a^2}{r^2}\right)^{\frac{1}{2}} e^{-\frac{25}{3}/(1+2a/r)}\right]. \quad . \quad . \quad . \quad (46)$$

Again, since $M' = 2M$, and $a' = 2a$, $\rho' = \frac{1}{4}\rho$, $\frac{w_0'}{\frac{1}{3}\rho'} = 4 \times \frac{w_0'}{\frac{1}{3}\rho} = 4 \times \cdot163$ by Table III., and $\frac{w_0'}{\frac{1}{3}\rho'} = \cdot65 = \frac{2}{3}$ nearly.

Thus, according to the second assumption, we have by (45)

$$3f(r) = \left(1 - \frac{2a}{r}\right) + \log\left(\frac{r}{2a}\right)^2, \quad \text{and, since } \frac{1}{\beta'^2} = 2\cdot78,$$

$$\left.\begin{aligned} \frac{3f(r)}{\beta'^2} &= 2\cdot78\left(1 - \frac{2a}{r}\right) + 2\cdot78\log\left(\frac{r}{2a}\right)^2, \\ \frac{3f(r)}{\beta'^2(1 - 4a^2/r^2)} &= \frac{2\cdot78}{1 + 2a/r} + \frac{2\cdot78}{1 - 4a^2/r^2}\log\left(\frac{r}{2a}\right)^2; \end{aligned}\right\}$$

and the law of density is

$$\frac{w}{\frac{1}{3}\rho} = \cdot163\left[e^{-3f(r)/\beta'^2} - \left(1 - \frac{4a^2}{r^2}\right)^{\frac{1}{2}} e^{-3f(r)/\beta'^2(1-4a^2/r^2)}\right]. \quad . \quad . \quad . \quad (47)$$

The values computed from these alternative formulæ (46) and (47) will be comparable with those in Table III.

In Table III. we have the value of $w/\frac{1}{3}\rho$ computed at distances $r/a = 2\cdot208$, $2\cdot463$, $2\cdot786$. The following short table gives the result extracted from Table III. for comparison with the values computed from (46) and (47) :—

| | $r/a =$ 2·0, | 2·208, | 2·463, | 2·786. |
|---|---|---|---|---|
| Convective equilib., $\frac{w}{\frac{1}{3}\rho} =$ | ·163, | ·092, | ·033, | 0. |
| First hypoth. (46), $\frac{w}{\frac{1}{3}\rho} =$ | ·163, | ·074, | ·033, | ·015. |
| Second hypoth. (47), $\frac{w}{\frac{1}{3}\rho} =$ | ·163, | ·071, | ·029, | ·011. |

It appears, therefore, that the results from the two hypotheses differ but little for some distance outside the region of collisions, and either line may be taken as near enough to the correct result. We see then that the effect of annulling collisions and allowing each body to describe an orbit is that the density at first falls off more rapidly than if the medium were in convective equilibrium, and that further away the density falls off less rapidly. At more remote distances the density would be found to tend to vary as the inverse square of this distance. Thus, the formulæ would make the mass of the system infinite. In other words, the existence of meteorites with nearly parabolic and hyperbolic orbits necessitates an infinite number, if the loss to the system is constantly made good by the supply.

The subject of this section is considered further, from a physical point of view, in the Summary at the end.

## § 14. *On the Kinetic Theory where the Meteorites are of all sizes.*

In an actual swarm of meteorites all sizes occur, for, even if this were not the case initially, inequality of size would soon arise through fractures. Hence, it becomes of interest to examine the kinetic theory on the hypothesis that the colliding bodies are of all possible sizes, grouped about some mean value according to some law of frequency.

If there be two sets of elastic spheres in such numbers that there are respectively $A$ and $B$ in unit volume, and if the mean squares of the velocities of the two are $\alpha^2$ and $\beta^2$ respectively, and if $a$ and $b$ are the radii of the spheres of the two sets, then it is proved that the number of collisions between them per unit time and volume is

$$2AB(a+b)^2\left[\tfrac{2}{3}\pi(\alpha^2+\beta^2)\right]^{\frac{1}{2}}\ *$$

We shall now change the notation, and for $a$ and $b$ write $s_1$ and $s_2$, and for $\alpha$ and $\beta$ write $u_1$ and $u_2$.

Then, if $\delta$ be the density of the spheres, their masses are $\frac{4}{3}\pi\delta s_1^3$ and $\frac{4}{3}\pi\delta s_2^3$.

The condition for the permanence of condition is that the spheres of all masses shall have the same mean kinetic energy. Hence, we refer the mass to a mean sphere of radius $\varsigma$, and the velocity to a square of velocity $V^2$.

Then

$$s_1^3u_1^2 = s_2^3u_2^2 = \varsigma^3V^2.$$

Thus, our formula may be written

$$2AB(s_1+s_2)^2\left[\left(\frac{\varsigma}{s_1}\right)^3+\left(\frac{\varsigma}{s_2}\right)^3\right]^{\frac{1}{2}}\left(\tfrac{2}{3}\pi V^2\right)^{\frac{1}{2}}.$$

* 'The Kinetic Theory of Gases,' by H. W. WATSON, p. 11.

But now suppose that there are spheres of all possible sizes, and that in unit volume the number whose radius lies between $s$ and $s + ds$ is

$$\frac{4n}{\sigma^3\sqrt{\pi}} s^2 e^{-s^2/\sigma^2} ds.^*$$

Since the integral of this from $\infty$ to 0 is $n$, it follows that $n$ is the number of spheres of all sizes in unit volume.

If $\rho$ be the total mass in unit volume, or the density of distribution,

$$\rho = \frac{4n}{\sqrt{\pi}} \cdot \int_0^\infty \tfrac{4}{3}\pi\delta s^3 \cdot \frac{s^2}{\sigma^3} e^{-s^2/\sigma^2} ds$$

$$= \frac{4n}{\sqrt{\pi}} \cdot \tfrac{4}{3}\pi\delta\sigma^3 \int_0^\infty x^5 e^{-x^2} dx$$

$$= \frac{4n}{\sqrt{\pi}} \cdot \tfrac{4}{3}\pi\delta\sigma^3 .$$

If $m$ be the mean mass, $m = \rho/n$; but $m = \frac{4}{3}\pi\delta\varsigma^3$; hence,

$$\varsigma^3 = \frac{4}{\sqrt{\pi}}\sigma^3,$$

and

$$\left(\frac{\varsigma}{s_1}\right)^3 + \left(\frac{\varsigma}{s_2}\right)^3 = \frac{4}{\sqrt{\pi}}\left[\left(\frac{\sigma}{s_1}\right)^3 + \left(\frac{\sigma}{s_2}\right)^3\right].$$

If the $A$ spheres of radii $s_1$ are those whose radii lie between $s_1$ and $s_1 + ds_1$, and the $B$ spheres of radii $s_2$ are those whose radii lie between $s_2$ and $s_2 + ds_2$,

$$A = \frac{4n}{\sqrt{\pi}}\left(\frac{s_1}{\sigma}\right)^2 e^{-s_1^2/\sigma^2}\frac{ds_1}{\sigma},$$

$$B = \frac{4n}{\sqrt{\pi}}\left(\frac{s_2}{\sigma}\right)^2 e^{-s_2^2/\sigma^2}\frac{ds_2}{\sigma}.$$

Hence, the formula for collisions between the $A$'s and $B$'s is

$$\frac{64n^2}{\pi^{\frac{5}{4}}} \cdot (\tfrac{2}{3}\pi V^2)^{\frac{1}{2}} \cdot (s_1 + s_2)^2 \left[\left(\frac{\sigma}{s_1}\right)^3 + \left(\frac{\sigma}{s_2}\right)^3\right]^{\frac{1}{2}} \frac{s_1^{\,2} s_2^{\,2}}{\sigma^4} e^{-(s_1^2+s_2^2)/\sigma^2} \frac{ds_1}{\sigma}\frac{ds_2}{\sigma},$$

or, if we write $x = s_1/\sigma$, $y = s_2/\sigma$, it is

$$\frac{64n^2}{\pi^{\frac{5}{4}}}(\tfrac{2}{3}\pi V^2)^{\frac{1}{2}} \sigma^2 (x+y)^2 (x^3+y^3)^{\frac{1}{2}} (xy)^{\frac{1}{2}} e^{-x^2-y^2} dx\, dy. \quad . \quad . \quad . \quad (48)$$

* If the spheres are grouped about a mean mass, instead of about a mean radius, according to a law of this kind, the subsequent integrals become very troublesome. Any law of the kind suffices for the discussion. If, however, I had foreseen the investigation of § 16, I should not have taken this law of frequency.

But

$$\sigma^2 = \frac{\pi^{\frac{1}{3}}}{2^{\frac{4}{3}}} \varsigma^2, \quad \text{and} \quad \frac{64}{\pi^{\frac{5}{4}}} \left(\tfrac{2}{3}\pi\right)^{\frac{1}{2}} \frac{\pi^{\frac{1}{3}}}{2^{\frac{4}{3}}} = \frac{32}{\pi^{\frac{5}{12}}} \cdot \frac{2^{\frac{1}{6}}}{3^{\frac{1}{2}}} \cdot$$

Hence, the number of collisions per unit time and volume between spheres whose radii range between $s_1$ and $s_1 + ds_1$, and others with radii between $s_2$ and $s_2 + ds_2$, is

$$\frac{32}{\pi^{\frac{5}{12}}} \frac{2^{\frac{1}{6}}}{3^{\frac{1}{2}}} \cdot V\varsigma^2 n^2 \,.\, (x+y)^2 (x^3+y^3)^{\frac{1}{2}} (xy)^{\frac{1}{2}} e^{-x^2-y^2} \, dx \, dy \,.$$

The number of collisions of a single sphere per unit time is $1/n$ of this, and, since $n = \rho/m$, we have for the collisions of a single sphere the factor $\dfrac{V\varsigma^2}{m/\rho}$ instead of $V\varsigma^2 n^2$.

Then the total number of collisions of all kinds in unit time, or the reciprocal of the mean free time, is the double integral of this from $\infty$ to 0.

For the purpose of carrying out the integration, we may conveniently, as an algebraic artifice, change from the rectangular axes $x$, $y$ to the polar coordinates $r$, $\theta$. Thus,

$$\int_0^\infty \int_0^\infty (x+y)^2 (x^3+y^3)^{\frac{1}{2}} (xy)^{\frac{1}{2}} e^{-x^2-y^2} \, dx \, dy$$
$$= \int_0^\infty r^{\frac{11}{2}} e^{-r^2} \, dr \int_0^{\frac{1}{2}\pi} (\sin\theta + \cos\theta)^{\frac{5}{2}} (1 - \sin\theta\cos\theta)^{\frac{1}{2}} (\sin\theta\cos\theta)^{\frac{1}{2}} \, d\theta.$$

Now, if we put $r = z^2$,

$$\int_0^\infty r^{\frac{11}{2}} e^{-r^2} \, dr = 2 \int_0^\infty z^{12} e^{-z^4} \, dz = 2 \cdot \frac{9\,.\,5\,.\,1}{4\,.\,4\,.\,4} \int_0^\infty e^{-z^4} \, dz.$$

For the transformation of the second integral, put

$$z = \cos\theta - \sin\theta,$$

and we find

$$\int_0^{\frac{1}{2}\pi} (\sin\theta + \cos\theta)^{\frac{5}{2}} (1 - \sin\theta\cos\theta)^{\frac{1}{2}} (\sin\theta\cos\theta)^{\frac{1}{2}} \, d\theta = \int_{-1}^{+1} \tfrac{1}{2} (2 - z^2)^{\frac{3}{4}} (1 - z^4)^{\frac{1}{2}} \, dz$$
$$= \int_0^{+1} (2 - z^2)^{\frac{3}{4}} (1 - z^4)^{\frac{1}{2}} \, dz.$$

Hence, the whole integral is

$$\tfrac{45}{32} \int_0^\infty e^{-z^4} \, dz \int_0^1 (2 - z^2)^{\frac{3}{4}} (1 - z^4)^{\frac{1}{2}} \, dz,$$

and the mean frequency of collision of a single ball per unit time is

$$\tfrac{15}{4} \cdot \frac{3^{\frac{1}{2}} \,.\, 2^{\frac{1}{6}}}{\pi^{\frac{5}{12}}} \frac{V (2\varsigma)^2}{m/\rho} \int_0^\infty e^{-z^4} \, dz \int_0^1 (2 - z^2)^{\frac{3}{4}} (1 - z^4)^{\frac{1}{2}} \, dz.$$

The second of these two integrals cannot, I think, be evaluated algebraically, but its value is easily found by quadratures. I find, then,

$$\int_0^1 (2 - z^2)^{\frac{3}{4}} (1 - z^4)^{\frac{1}{2}} \, dz = 1 \cdot 2999.$$

The former of the two integrals may be evaluated as follows :—

Let

$$I = \int_0^\infty e^{-x^4} \, dx,$$

then,

$$\begin{aligned} 4I^2 &= 4 \int_0^\infty \int_0^\infty e^{-x^4 - y^4} \, dx \, dy \\ &= 4 \int_0^\infty \int_0^{\frac{1}{2}\pi} e^{-r^4 (1 - \frac{1}{2} \sin^2 2\theta)} \, r \, dr \, d\theta \\ &= \int_0^\infty \int_0^\pi e^{-z^2 (1 - \frac{1}{2} \sin^2 \phi)} \, dz \, d\phi \\ &= 2 \int_0^{\frac{1}{2}\pi} \int_0^\infty \frac{e^{-t^2}}{(1 - \frac{1}{2} \sin^2 \phi)} \, dt \, d\phi \\ &= \pi^{\frac{1}{2}} \int_0^{\frac{1}{2}\pi} \frac{d\phi}{(1 - \frac{1}{2} \sin^2 \phi)^{\frac{1}{2}}} \\ &= \pi^{\frac{1}{2}} F(45^\circ), \end{aligned}$$

where $F$ is the complete elliptic integral with modulus sin 45°.

Hence,

$$I = \tfrac{1}{2} \pi^{\frac{1}{4}} F^{\frac{1}{2}}.*$$

We thus have the frequency of collision given by

$$\tfrac{15}{8} \cdot \frac{3^{\frac{1}{4}} . 2^{\frac{1}{4}}}{\pi^{\frac{1}{4}}} \cdot F^{\frac{1}{2}} \cdot 1 \cdot 2999 \cdot \frac{V (2\varsigma)^2}{m/\rho} \cdot$$

Now, LEGENDRE's Tables give

$$\log F = \cdot 2681272,$$

with which value we easily find for $T$ the mean free time, or $1/T$ the frequency,

$$\frac{1}{T} = 5 \cdot 3318 \frac{V (2\varsigma)^2}{m/\rho} = \tfrac{16}{3} \frac{V (2\varsigma)^2}{m/\rho} \text{ nearly.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

If $1/T_0$ be the frequency of collision when the spheres are all of the same size and mass $\varsigma$ and $m$, and are agitated with mean square of velocity $V^2$, we have, by the ordinary theory,

$$\frac{1}{T_0} = 4 \sqrt{\frac{\pi}{3}} \cdot \frac{V (2\varsigma)^2}{m/\rho} = 4 \cdot 0935 \frac{V (2\varsigma)^2}{m/\rho} \cdot \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

* I owe this to Mr. FORSYTH, and the result verifies an evaluation by quadratures which I had made.

It follows, therefore, that in our case collisions are more frequent than if the balls were all of the same size in about the proportion of 4 to 3.

In order to find the mean free path, we require to find the mean velocity.

If $u^2$ be the mean square of the velocity for any size $s$, the proportion of all the spheres of that size which move with velocities lying between $v$ and $v + dv$ is

$$\frac{4}{\sqrt{\pi}} y^2 e^{-y^2} dy,$$

where $y^2 = 3v^2/2u^2$.

But the number of spheres of size between $s$ and $s + ds$, in unit volume, is

$$\frac{4n}{\sqrt{\pi}} x^2 e^{-x^2} dx,$$

where $x = s/\sigma$.

Hence, the mean velocity $U$ is given by

$$U = \frac{16}{\pi} \int_0^\infty \int_0^\infty v x^2 y^2 e^{-x^2-y^2} dx\, dy.$$

Now,

$$v = \sqrt{\tfrac{2}{3}} \,.\, uy, \text{ and } s^3 u^2 = \mathfrak{s}^3 V^2, \text{ or } x^3 u^2 = \left(\frac{\mathfrak{s}}{\sigma}\right)^3 V^2 = \frac{4}{\sqrt{\pi}} V^2,$$

so that

$$u = \frac{2}{\pi^{\frac{1}{4}}} x^{-\frac{3}{2}} V, \text{ and } v = \frac{2\sqrt{2}}{\pi^{\frac{1}{4}}\sqrt{3}} x^{-\frac{3}{2}} y V.$$

Therefore,

$$U = \frac{32\sqrt{2}}{\pi^{\frac{5}{4}}\sqrt{3}} V \int_0^\infty \int_0^\infty x^{\frac{1}{2}} y^3 e^{-x^2-y^2} dx\, dy.$$

But

$$\int_0^\infty y^3 e^{-y^2} dy = \tfrac{1}{2}, \text{ and } \int_0^\infty x^{\frac{1}{2}} e^{-x^2} dx = 2\int_0^\infty z^2 e^{-z^4} dz;$$

therefore,

$$U = \frac{32\sqrt{2}}{\pi^{\frac{5}{4}}\sqrt{3}} V \int_0^\infty z^2 e^{-z^4} dz.$$

This integral may be evaluated as follows :—

Let

$$J = \int_0^\infty x^2 e^{-x^4} dx,$$

$$\begin{aligned}
4J^2 &= 4\int_0^\infty \int_0^\infty x^2 y^2 e^{-x^4-y^4} dx\, dy \\
&= \int_0^\infty \int_0^{\frac{1}{2}\pi} r^4 \sin^2 2\theta \, e^{-r^4(1-\frac{1}{2}\sin^2 2\theta)} r\, dr\, d\theta \\
&= \tfrac{1}{4}\int_0^\infty \int_0^{\pi} z^2 \sin^2 \phi \, e^{-z^2(1-\frac{1}{2}\sin^2\phi)} dz\, d\phi \\
&= \tfrac{1}{2}\int_0^\infty \int_0^{\frac{1}{2}\pi} \frac{\sin^2\phi}{(1-\frac{1}{2}\sin^2\phi)^{\frac{3}{2}}} t^2 e^{-t^2} dt\, d\phi \\
&= \tfrac{1}{4}\pi^{\frac{1}{2}} \int_0^{\frac{1}{2}\pi} \frac{\frac{1}{2}\sin^2\phi}{(1-\frac{1}{2}\sin^2\phi)^{\frac{3}{2}}} d\phi \\
&= \tfrac{1}{4}\pi^{\frac{1}{2}} \left[\int_0^{\frac{1}{2}\pi} \frac{d\phi}{(1-\frac{1}{2}\sin^2\phi)^{\frac{3}{2}}} - F\right].
\end{aligned}$$

Now,

$$\int_0^{\frac{1}{2}\pi} \frac{d\phi}{(1 - k^2 \sin^2 \phi)^{\frac{3}{2}}} = \frac{E}{k'^2},$$

and in the present case $k^2 = k'^2 = \frac{1}{2}$.

Hence,

$$J = \tfrac{1}{4}\pi^{\frac{1}{4}} [2E - F]^{\frac{1}{2}} *,$$

where $E$ and $F$ are the complete elliptic integrals with modulus sin 45°.

In LEGENDRE's Tables, we find

$$E = 1{\cdot}350644, \quad F = 1{\cdot}854075, \quad \text{and } 2E - F = {\cdot}847213.$$

Then,

$$\frac{U}{V} = \frac{8}{\pi} \sqrt{\tfrac{2}{3}} \sqrt{(2E - F)} = 1{\cdot}91377.$$

The mean free path

$$L = UT = 1{\cdot}9138\ VT = \frac{1{\cdot}9138}{5{\cdot}3318} \frac{m/\rho}{(2\varsigma)^2},$$

and thus

$$L = \frac{1}{2{\cdot}786} \frac{m/\rho}{(2\varsigma)^2}. \quad \quad (51)$$

If the spheres had all been of the same size, we should have had

$$L_0 = \frac{m/\rho}{\pi (2\varsigma)^2 \sqrt{2}} = \frac{1}{4{\cdot}44} \frac{m/\rho}{(2\varsigma)^2}. \quad \quad (52)$$

Hence, finally from (49) to (52), if there be a number of spherical meteorites, of uniform density, of all sizes with radii grouped about a mean radius according to the law of error, and if $S$ be the *diameter* of the meteorite of mean mass $m$, and $\rho$ be the density of the distribution of meteorites in space, and $\frac{1}{2} m V^2$ their mean kinetic energy of agitation, then the mean free path $L$, mean free time $T$, and mean velocity $U$ are given by

$$\left.\begin{aligned} L &= \frac{1}{2{\cdot}786} \frac{m/\rho}{S^2} = \tfrac{5}{14} \frac{m/\rho}{S^2} \text{ nearly,} \\ T &= \frac{1}{5{\cdot}332} \frac{m/\rho}{VS^2} = \tfrac{3}{16} \frac{m/\rho}{VS^2} \text{ nearly,} \\ U &= 1{\cdot}9138\ V = 2V \text{ nearly.} \end{aligned}\right\} \quad \quad (53)$$

Also the mean free path is about $\frac{7}{11}$ths, and the mean free time about $\frac{3}{4}$ of that which would have held if the meteorites had all been of the same size $m$ and had had the same mean kinetic energy $\frac{1}{2} m V^2$.

* I owe this to Mr. FORSYTH.

§ 15. *On the Variation of Mean Frequency of Collision and Mean Free Path for the several sizes of balls.*

Each size of ball has its own mean frequency of collision and mean free path, and it is well to trace how the total means evaluated in the last section are made up.

We have already seen in (48) that (substituting for $\sigma$ its value in terms of $\varsigma$) the number of collisions per unit time and volume between balls of sizes $s$ to $s + ds$ and balls of sizes $s'$ to $s' + ds'$ is

$$\frac{64n^2}{\pi^{\frac{5}{4}}} \left(\tfrac{2}{3}\,\pi V^2\right)^{\frac{1}{2}} \cdot \frac{\pi^{\frac{1}{3}}}{2^{\frac{4}{3}}}\, \varsigma^2 (x + y)^2 (x^3 + y^3)^{\frac{1}{2}} (xy)^{\frac{1}{2}} e^{-x^2-y^2}\, dx\, dy,$$

where $x = s/\sigma$, $y = s'/\sigma$.

But the number of balls of size $s$ to $s + ds$ in unit volume is

$$\frac{4n}{\sqrt{\pi}}\, x^2 e^{-x}\, dx.$$

Hence, the mean frequency of collision for a ball of size $s$ with all others is

$$\frac{\pi^{\frac{1}{2}}}{4n} \cdot \frac{64n^2}{\pi^{\frac{5}{4}}} \left(\tfrac{2}{3}\,\pi V^2\right)^{\frac{1}{2}} \frac{\pi^{\frac{1}{3}}}{2^{\frac{4}{3}}}\, \varsigma^2 \int_0^\infty (x + y)^2 (x^3 + y^3)^{\frac{1}{2}} x^{-\frac{3}{2}} y^{\frac{1}{2}} e^{-y^2}\, dy.$$

Now,

$$x^{-\frac{3}{2}} = \left(\frac{\sigma}{s}\right)^{\frac{3}{2}} = \tfrac{1}{2}\pi^{\frac{1}{4}} \cdot \left(\frac{\varsigma}{s}\right)^{\frac{3}{2}}.$$

Therefore, if we write $1/\tau$ for the frequency of collision of a ball of size $s$ with all others, we have

$$\frac{1}{\tau} = \frac{2^{\frac{1}{6}} \cdot \pi^{\frac{1}{3}}}{3^{\frac{1}{2}}} \cdot \frac{V (2\varsigma)^2}{m/\rho} \left(\frac{\varsigma}{s}\right)^{\frac{3}{2}} \int_0^\infty (x + y)^2 (x^3 + y^3)^{\frac{1}{2}} y^{\frac{1}{2}} e^{-y^2}\, dy.$$

Now, the mean frequency for all sizes is given by

$$\frac{1}{T} = 5{\cdot}3318 \cdot \frac{V (2\varsigma)^2}{m/\rho}.$$

Hence,

$$\frac{T}{\tau} = \frac{1}{5{\cdot}3318} \cdot \frac{2^{\frac{1}{6}} \cdot \pi^{\frac{1}{3}}}{3^{\frac{1}{2}}} \cdot \left(\frac{\varsigma}{s}\right)^{\frac{3}{2}} \int_0^\infty (x + y)^2 (x^3 + y^3)^{\frac{1}{2}} y^{\frac{1}{2}} e^{-y^2}\, dy.$$

$$= {\cdot}1780 \cdot \left(\frac{\varsigma}{s}\right)^{\frac{3}{2}} \int_0^\infty (x + y)^2 (x^3 + y^3)^{\frac{1}{2}} y^{\frac{1}{2}} e^{-y^2}\, dy. \quad . \quad . \quad . \quad . \quad . \quad (54)$$

The integral involved here cannot in general be determined algebraically; but, if $x$ be very small, or very great, we can find an approximate value for it.

If $x$ be very small, the integral becomes

$$\int_0^\infty y^4 e^{-y^2}\, dy = \tfrac{3}{8}\sqrt{\pi}, \qquad \text{and} \qquad \frac{T}{\tau} = {\cdot}118 \left(\frac{\varsigma}{s}\right)^{\frac{3}{2}}.$$

If $x$ be very large, the integral becomes

$$x^{\frac{7}{2}} \int_0^\infty y^{\frac{1}{2}} e^{-y^2} dy = 2x^{\frac{7}{2}} \int_0^\infty z^2 e^{-z^4} dz.$$

Now,

$$x^{\frac{7}{2}} = \left(\frac{s}{\sigma}\right)^{\frac{7}{4}} = \frac{2^{\frac{7}{3}}}{\pi^{\frac{7}{12}}} \left(\frac{s}{\varsigma}\right)^{\frac{7}{2}}, \qquad \text{and} \qquad \int_0^\infty z^2 e^{-z^4} dz = \tfrac{1}{4}\pi^{\frac{1}{4}} (2E - F)^{\frac{1}{2}}.$$

Therefore, the integral becomes

$$\frac{2^{\frac{4}{3}}}{\pi^{\frac{1}{3}}} (2E - F)^{\frac{1}{2}} \left(\frac{s}{\varsigma}\right)^{\frac{1}{2}},$$

and with the known values of $E$ and $F$ this gives us

$$\frac{T}{\tau} = \cdot 282 \left(\frac{s}{\varsigma}\right)^2.$$

For intermediate values of $s$ recourse must be had to quadratures for evaluating the integral. I have therefore determined, by a rough numerical process, sufficient values of the integral to render possible the drawing of a curve for the values of $T/\tau$ for all values of $s$. The following table gives the results for the integral $\int_0^\infty (x+y)^2 (x^3+y^3)^{\frac{1}{2}} y^{\frac{1}{2}} e^{-y^2} dy$, which may be denoted by $K$:—

| | $K$ |
|---|---|
| $s = \frac{1}{2}\varsigma$ | 1·71 , |
| $s = \frac{3}{4}\varsigma$ | 2·90 , |
| $s = \varsigma$ | 4·94 , |
| $s = \frac{3}{2}\varsigma$ | 12·97 , |
| $s = 2\varsigma$ | 28·75 . |

If these values be introduced in the formula

$$\frac{T}{\tau} = \cdot 1780\, K \left(\frac{\varsigma}{s}\right)^{\frac{3}{2}},$$

we obtain

| | $T/\tau$ |
|---|---|
| $s = \frac{1}{2}\varsigma$ | ·86 , |
| $s = \frac{3}{4}\varsigma$ | ·80 , |
| $s = \varsigma$ | ·88 , |
| $s = \frac{3}{2}\varsigma$ | 1·26 , |
| $s = 2\varsigma$ | 1·81 . |

These values are used for forming the curve, entitled "frequency of collision," in fig. 1 below, and they are supplemented by the values found above for $T/\tau$, in the case where $s/\varsigma$ is either very small or very large.

The frequency becomes infinite when the balls are infinitely small, because of the infinite velocity with which they move, and again infinite for infinitely large balls, because of their infinite size. But it must be remembered that there are infinitely few balls of these two limiting sizes.

We have now to consider the mean free path, say $\lambda$, for the several sizes.

If $u^2$ be the mean square of velocity for the size $s$, the mean velocity for that size is $u\sqrt{(8/3\pi)}$, by the ordinary kinetic theory.

From the constancy of mean kinetic energy for all sizes, we have

$$s^3 u^2 = \varsigma^3 V^2,$$

so that the mean velocity for size $s$ is

$$V(\varsigma/s)^{\frac{3}{2}}\sqrt{(8/3\pi)}.$$

But, if $U$ be the mean velocity, and $L$ the mean free path, and $T$ the mean free time for all sizes together, we have

$$V = \frac{U}{1\cdot9138} = \frac{1}{1\cdot9138}\,\frac{L}{T}.$$

Therefore, the mean velocity for size $s$ is

$$\frac{\sqrt{(8/3\pi)}}{1\cdot9138}\left(\frac{\varsigma}{s}\right)^{\frac{3}{2}}\frac{L}{T} = \cdot4815\left(\frac{\varsigma}{s}\right)^{\frac{3}{2}}\frac{L}{T}\,.$$

But the mean velocity for size $s$ is $\lambda/\tau$; hence,

$$\frac{\lambda}{\mathrm{L}} = \cdot4815\left(\frac{\varsigma}{s}\right)^{\frac{3}{2}}\frac{\tau}{\mathrm{T}} = \frac{4815}{1780}\cdot\frac{1}{\mathrm{K}}$$

$$= \frac{2\cdot705}{\mathrm{K}}\,.$$

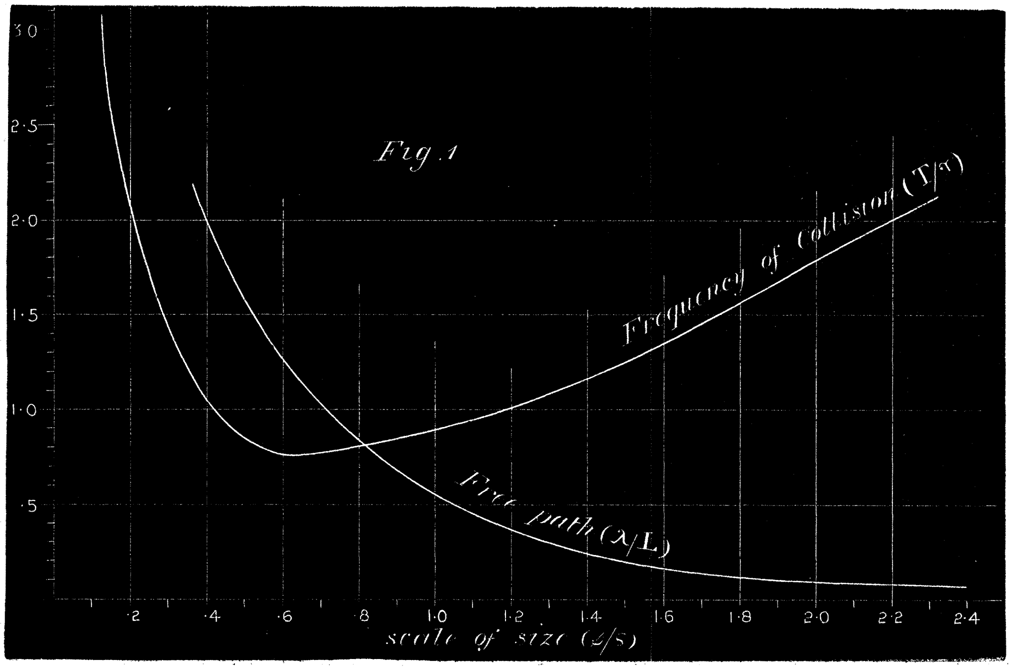
When $s$ is very small, we find $\lambda/\mathrm{L} = 4$, and, when $s$ is very large, $\lambda/\mathrm{L} = 1\cdot7\,(\varsigma/s)^{\frac{7}{2}}$. Thus, for small values of $s$, the mean free path reaches a constant limit 4, and for large values it becomes infinitely small.

The intermediate values, sufficient for drawing a curve, are given in the following short table:—

| | $\lambda/\mathrm{L}$. |
|---|---|
| $s = \frac{1}{2}\varsigma$ | 1·58. |
| $s = \frac{3}{4}\varsigma$ | ·93. |
| $s = \varsigma$ | ·55. |
| $s = \frac{3}{2}\varsigma$ | ·21. |
| $s = 2\varsigma$ | ·09. |

These values are set out in the annexed figure in the curve marked "free path," and are supplemented by the values found above for small and large values of $s$. The constant limit 4 falls outside the figure. The horizontal portion of the curve is asymptotic to the $s$-axis.

Fig. 1.




No immediate use is made of these conclusions, but it was proper to examine this point in the theory.

## § 16. *On the Sorting of Meteorites according to size and its Results.*

It is a well-known result of the kinetic theory of gases, that if a number of different gases co-exist, each gas has the same density as though it alone existed, and was subject to the resultant forces of the system; also the mean kinetic energy of agitation of each gas is the same. From this it follows that the elasticity of each gas is inversely proportional to the mass of its molecule.

Carrying on this conclusion to meteorites, we see that the elasticity of the gas formed by large meteorites is less than that for small; and, hence, there is a greater concentration of large meteorites towards the centre, and there will be a sorting according to size. The object of this section is to investigate this point.

In §§ 14 and 15, the laws of a kinetic theory were investigated when the gas consisted of molecules of all masses, grouped, according to a law of frequency, about a certain mean radius, and molecules of infinite mass were considered to be admissible, with, of course, infinite rarity. Now, if we were to continue to use that law of frequency of masses in the present investigation, we should find, as an analytical result, that the mean mass in the centre of the swarm becomes infinite. The existence of very large meteorites in sufficient numbers to give statistical constancy in a volume which is not a considerable fraction of the volume of the whole swarm is physically improbable. We shall, therefore, treat the case best by absolutely excluding very large masses. When such masses occur, they must not be treated statistically; this is a question which I hope to consider in a future paper. Had I foreseen this conclusion when the investigations of the last two sections were carried out, a different law of frequency of mass would have been assumed. But the results of those sections are amply sufficient to indicate the conclusions which would have been reached with another law of frequency, and, therefore, it does not seem worth while to recompute the results by means of a fresh series of laborious quadratures.

Any law of frequency would suffice for our purpose which excludes masses greater than a certain limit and rises to a maximum for a certain mean mass. For the present, I do not specify that law precisely, but merely assume that at some radius, which may conveniently be taken as that of the isothermal sphere, where $r = a$, the number of meteorites whose masses lie between $x$ and $x + \delta x$ is $f(x)\,\delta x$; it is also assumed that $x$ may range from M* to zero.

The meteorites whose masses range from $x$ to $x + \delta x$ may be deemed to constitute a gas. Suppose that at radius $r$ the number of its molecules per unit volume is $\delta n$, its density $\delta w$, its pressure $\delta p$, and let the same symbols, with suffix 0, denote the same things at radius $a$. Since all the partial gases are in the permanent state, they all have the same mean kinetic energy of agitation, equal to $\frac{1}{2}h$, suppose. Throughout the isothermal sphere, this $h$ is constant, and equal, say, to $h_0$, but varies with the radius in the adiabatic layer over it. It follows, therefore, that the mean square of the velocity of the particular partial gas $x$ to $x + \delta x$ is equal to $h/x$, and the relation between $\delta p$ and $\delta w$ is

$$\delta p = \tfrac{1}{3}\frac{h}{x}\,\delta w.$$

Let $-\chi$ be the excess of the gravitation potential of the *whole* swarm at radius $r$ above its value at radius $a$.

Then, since each partial gas behaves as though it existed by itself, the equation of hydrostatic equilibrium of the partial gas $x$ to $x + \delta x$ is

$$\frac{1}{\delta w}\frac{d\,\delta p}{dr} + \frac{d\chi}{dr} = 0.$$

The investigation must now divide into two, according as whether we are considering the isothermal sphere or the adiabatic layer.

*The Isothermal Sphere.*

Here we have $h$ a constant and equal to $h_0$, and $\delta p$ varies as $\delta w$, so that

* This M is not to be confused with $M$, the mass of the isothermal sphere.

$$\tfrac{1}{3}\frac{h_0}{x}\log\frac{\delta w}{\delta w_0} = -\chi,$$

or

$$\frac{\delta w}{\delta w_0} = e^{-3\chi x/h_0}.$$

Now it is obvious that $\delta n/\delta n_0 = \delta w/\delta w_0$; and, therefore,

$$\delta n = e^{-3\chi x/h_0}\,\delta n_0.$$

But, by the definition of $f(x)$,

$$\delta n_0 = f(x)\,\delta x\,;$$

hence,

$$\delta n = e^{-3\chi x/h_0} f(x)\,\delta x. \qquad (55)$$

This is the law of frequency of mass $x$ to $x + \delta x$ at radius $r$.

Now, if $m$, $m_0$ be the mean masses at radii $r$ and $a$ respectively,

$$m = \frac{\int_0^{\mathrm{M}} x\,e^{-3\chi x/h_0} f(x)\,dx}{\int_0^{\mathrm{M}} e^{-3\chi x/h_0} f(x)\,dx}\,; \qquad (56)$$

and, if we put $\chi = 0$, we obtain $m_0$ from the same formula.

It is also clear that, if $w$ be the total density of the swarm at radius $r$,

$$w = \int x\,dn = \int_0^{\mathrm{M}} x\,e^{-3\chi x/h_0} f(x)\,dx. \qquad (57)$$

By the definition of $\chi$, and in consequence of the supposed spherical arrangement of matter, we have

$$\chi = \int_a^r \frac{1}{r^2}\left(\int_o^r 4\pi\mu w r^2 dr\right) dr.$$

If this value were substituted in (57), we should obtain a very complicated differential equation to determine $w$, the solution of which is hopelessly difficult. We may, however, assume without much error that the $w$ in the integral expressing $\chi$ is the density of meteorites, all of which are of the same size $m'$, and which are agitated with mean kinetic energy $\frac{1}{2}h_0$. If this density be written $\mathrm{w}$, we then clearly have

$$\chi = -\frac{h_0}{3m'}\log\frac{\mathrm{w}}{\mathrm{w}_0}.$$

The values of $\mathrm{w}$ and $\mathrm{w}_0$ may be extracted from Table III. of solutions in § 6.

Then we have

$$-\frac{3\chi x}{h_0} = \frac{x}{m'} \log \frac{w}{w_0} = qx, \text{ suppose,}$$

where $q$ is rigorously equal to $-3\chi/h_0$; but for computing the approximate value $(1/m') \log (w/w_0)$ is to be employed.

In order to proceed to the evaluation of the mean mass at various distances, we must assume some form for $f(x)$.

I assume, then, that

$$f(x) = \frac{6n_0}{M^3} x (M - x).$$

It is easy to show that

$$\int_0^M f(x)\, dx = n_0, \quad \text{and} \quad \frac{1}{n_0} \int_0^M x f(x)\, dx = \tfrac{1}{2}M.$$

Hence, the mean mass $m_0 = \frac{1}{2}M$, and the maximum frequency is for masses equal to $m_0$.

Then, by (56), we have for the mean mass at radius $r$

$$m = \frac{\int_0^M x^2 (M - x)\, e^{qx} dx}{\int_0^M x (M - x)\, e^{qx} dx}.$$

But

$$\left.\begin{aligned} \int_0^M x^2 (M - x)\, e^{qx} dx &= \frac{1}{q^4} [e^{Mq} (M^2 q^2 - 4Mq + 6) - 2 (Mq + 3)], \\ \int_0^M x (M - x)\, e^{qx} dx &= \frac{1}{q^3} [e^{Mq} (Mq - 2) + (Mq + 2)]. \end{aligned}\right\} \quad . \quad . \quad (58)$$

It may be remarked that, if $Mq$ be treated as small, we have the first of these integrals equal to $\frac{1}{12}M^4 (1 + \frac{3}{5}Mq)$, and the second equal to $\frac{1}{6}M^3 (1 + \frac{1}{2}Mq)$, and the ratio of the first to the second is $\frac{1}{2}M (1 + \frac{1}{10}Mq)$.

In order to evaluate $m$, we proceed to introduce the approximate value for $q$.

Now,

$$q = \frac{1}{m'} \log \frac{w}{w_0}, \quad \text{and} \quad e^{Mq} = \left(\frac{w}{w_0}\right)^{M/m'};$$

then, writing for brevity,

$$P = \log \left(\frac{w}{w_0}\right)^{M/m'},$$

we have

$$\frac{m}{\frac{1}{2}M} = \frac{2}{P} \cdot \frac{\left(\frac{w}{w_0}\right)^{M/m'} (P^2 - 4P + 6) - 2 (P + 3)}{\left(\frac{w}{w_0}\right)^{M/m'} (P - 2) + (P + 2)}. \quad . \quad . \quad . \quad . \quad . \quad (59)$$

Also, if $P$ be small, the approximate result is

$$\frac{m}{\frac{1}{2}\mathrm{M}} = 1 + \tfrac{1}{10}P.$$

Before proceeding to give numerical values for the fall of mean mass as we proceed outwards from the centre of the isothermal sphere, we must consider

*The Adiabatic Layer.*

In this case we assume, as before, that the ratio of the two specific heats is $1\frac{2}{3}$, and we therefore have for the relationship between $\delta p$ and $\delta w$ at radius $r$,

$$\frac{\delta p}{\delta p_0} = \left(\frac{\delta w}{\delta w_0}\right)^{\frac{5}{3}}.$$

Hence,

$$\frac{1}{\delta w}\frac{d\,\delta p}{dr} = \tfrac{5}{6}\left[\frac{3\,\delta p_0}{\delta w_0}\right]\frac{d}{dr}\left(\frac{\delta w}{\delta w_0}\right)^{\frac{2}{3}}.$$

But, since $\delta p_0$, $\delta w_0$ apply to the radius $a$ where $h = h_0$, a constant,

$$\frac{3\,\delta p_0}{\delta w_0} = \frac{h_0}{x}.$$

Thus, in the adiabatic layer the equation of hydrostatic equilibrium is

$$\tfrac{5}{6}\frac{h_0}{x}\frac{d}{dr}\left(\frac{\delta w}{\delta w_0}\right)^{\frac{2}{3}} + \frac{d\chi}{dr} = 0,$$

whence,

$$\chi = \tfrac{5}{6}\frac{h_0}{x}\left(1 - \left(\frac{\delta w}{\delta w_0}\right)^{\frac{2}{3}}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

or

$$\delta w = \delta w_0\left[1 - \frac{6\chi x}{5h_0}\right]^{\frac{3}{2}}.$$

The investigation now follows a line parallel to that taken before.

We have $\delta n/\delta n_0 = \delta w/\delta w_0$, and $\delta n_0 = f(x)\,dx$, so that

$$\delta n = \left(1 - \frac{6\chi x}{5h_0}\right)^{\frac{3}{2}} f(x)\,\delta x.$$

This is the law of frequency of masses lying between $x$ and $x + \delta x$ at radius $r$.

As $\delta n$ can never be negative, we see that there can be no mass greater than $\frac{5}{6}h_0/\chi$; and, if M be the greatest positive value of the expression $f(x)$, there can be no mass greater than the smaller of $\frac{5}{6}h_0/\chi$ or M.

Thus, if $m$ be the mean mass at radius $r$,

$$m = \frac{\int_0^\alpha x\left(1 - \frac{6\chi x}{5h_0}\right)^{\frac{3}{2}} f(x)\,dx}{\int_0^\alpha \left(1 - \frac{6\chi x}{5h_0}\right)^{\frac{3}{2}} f(x)\,dx}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

where $\alpha$ is the smaller of $\frac{5}{6}h_0/\chi$ and M.

If we put $\chi = 0$ in (61), we obtain $m_0$, the mean mass at radius $a$.

It is clear also that, if $w$ be the total density of the swarm at radius $r$,

$$w = \int x\,dn = \int_0^\alpha x\left(1 - \frac{6\chi x}{5h_0}\right)^{\frac{3}{2}} f(x)\,dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

By definition of $\chi$, and in consequence of the supposed spherical arrangement of matter, we have

$$\chi = \int_a^r \frac{1}{r^2}\left(\int_0^r 4\pi\mu w r^2\,dr\right)dr.$$

If this value were substituted in (62), we might obtain a complicated differential equation for $w$. It is clear, however, that an adequate approximation may be obtained by assuming that the $w$ in the integral expressing $\chi$ is the density of meteorites, all of which are of the same size $m'$, arranged in a layer in convective equilibrium, and with kinetic energy of agitation at the limit $r = a$ equal to $\frac{1}{2}h_0$.

If this density be written w, and if $\mathrm{v}^2$ be the mean square of velocity of agitation at radius $r$, we have, by (60), and in consequence of the relationship $(\mathrm{w}/\mathrm{w}_0)^{\frac{2}{3}} = (\mathrm{v}/\mathrm{v}_0)^2$,

$$\chi = \tfrac{5}{6}\frac{h_0}{m'}\left(1 - \left(\frac{\mathrm{w}}{\mathrm{w}_0}\right)^{\frac{2}{3}}\right) = \tfrac{5}{6}\mathrm{v}_0^2\left(1 - \frac{\mathrm{v}^2}{\mathrm{v}_0^2}\right),$$

and

$$\tfrac{6}{5}\frac{\chi x}{h_0} = \frac{x}{m'}\left(1 - \frac{\mathrm{v}^2}{\mathrm{v}_0^2}\right).$$

Let

$$\frac{1}{\beta} = \frac{1}{m'}\left(1 - \frac{\mathrm{v}^2}{\mathrm{v}_0^2}\right)$$

for brevity; then, adopting the law of frequency $f(x) = \frac{6\,n_0}{\mathrm{M}^3}x(\mathrm{M} - x)$, as before, we have for the mean mass at radius $r$

$$m = \frac{\int_0^\alpha x^2(\mathrm{M} - x)\left(1 - \frac{x}{\beta}\right)^{\frac{3}{2}} dx}{\int_0^\alpha x(\mathrm{M} - x)\left(1 - \frac{x}{\beta}\right)^{\frac{3}{2}} dx}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

where $\alpha$ is equal to the smaller of M and $\beta$.

The solution now becomes different according as M or $\beta$ is the smaller.

First, suppose M is the smaller. Then the limits of integration are M and 0.

If we put

$$z = 1 - \frac{x}{\beta}.$$

$$x^n \left(1 - \frac{x}{\beta}\right)^{\frac{3}{2}} dx = -\beta^{n+1} z^{\frac{3}{2}} \left(1 - nz + \frac{n \,.\, n-1}{1 \,.\, 2} z^2 - \ldots\right) dz,$$

so that the numerator and denominator of $m$ are easily integrable.

If now we write

$$Q = 1 - \frac{\mathrm{M}}{\beta},$$

$$\int_0^{\mathrm{M}} x^2 (\mathrm{M} - x) \left(1 - \frac{x}{\beta}\right)^{\frac{3}{2}} dx = 2\beta^4 \left[\tfrac{1}{5}\left(\frac{\mathrm{M}}{\beta} - 1\right)(1 - Q^{\frac{5}{2}}) - \tfrac{1}{7}\left(2\frac{\mathrm{M}}{\beta} - 3\right)(1 - Q^{\frac{7}{2}}) + \tfrac{1}{9}\left(\frac{\mathrm{M}}{\beta} - 3\right)(1 - Q^{\frac{9}{2}}) + \tfrac{1}{11}(1 - Q^{\frac{11}{2}})\right]$$

$$= 2\beta^4 \left[\frac{8}{7 \,.\, 9 \,.\, 11} - \frac{8}{5 \,.\, 7 \,.\, 9} Q + \frac{2}{5 \,.\, 7} Q^{\frac{7}{2}} - \frac{4}{7 \,.\, 9} Q^{\frac{9}{2}} + \frac{2}{9 \,.\, 11} Q^{\frac{11}{2}}\right],$$

$$\int_0^{\mathrm{M}} x (\mathrm{M} - x) \left(1 - \frac{x}{\beta}\right)^{\frac{3}{2}} dx = 2\beta^3 \left[\tfrac{1}{5}\left(\frac{\mathrm{M}}{\beta} - 1\right)(1 - Q^{\frac{5}{2}}) - \tfrac{1}{7}\left(\frac{\mathrm{M}}{\beta} - 2\right)(1 - Q^{\frac{7}{2}}) - \tfrac{1}{9}(1 - Q^{\frac{9}{2}})\right]$$

$$= 2\beta^3 \left[\frac{2}{7 \,.\, 9} - \frac{2}{5 \,.\, 7} Q + \frac{2}{5 \,.\, 7} Q^{\frac{7}{2}} - \frac{2}{7 \,.\, 9} Q^{\frac{9}{2}}\right].$$

Then, since $\beta = (1 - Q)/M$, we have

$$\frac{m}{M} = \frac{\frac{4}{11} - \frac{4}{5} Q + \frac{9}{5} Q^{\frac{7}{2}} - 2Q^{\frac{9}{2}} + \frac{7}{11} Q^{\frac{11}{2}}}{(1 - Q)(1 - \frac{9}{5} Q + \frac{9}{5} Q^{\frac{7}{2}} - Q^{\frac{9}{2}})} \,. \qquad (64)$$

This expression has a high order of indeterminateness when $Q = 1$, but I find that when $Q$ is nearly equal to unity

$$\frac{m}{\mathrm{M}} = \tfrac{1}{2}\left[1 - \tfrac{3}{10}(1 - Q^{\frac{1}{2}})\right] \text{ nearly.} \qquad (65)$$

Thus, the mean mass is $\frac{1}{2}$M where $r = a$, which we know to be correct.

Secondly, suppose that $\beta$ is smaller than M. Then effecting the integrations in the same manner as before, we have

$$\int_0^{\beta} x^2 (\mathrm{M} - x) \left(1 - \frac{x}{\beta}\right)^{\frac{3}{2}} dx = 2\beta^4 \left[\tfrac{1}{5}\left(\frac{\mathrm{M}}{\beta} - 1\right) - \tfrac{1}{7}\left(\frac{2\mathrm{M}}{\beta} - 3\right) + \tfrac{1}{9}\left(\frac{\mathrm{M}}{\beta} - 3\right) + \tfrac{1}{11}\right]$$

$$= \frac{2 \,.\, 8}{5 \,.\, 7 \,.\, 9} \beta^4 \left(\frac{\mathrm{M}}{\beta} - \tfrac{6}{11}\right),$$

$$\int_0^\beta x(\mathrm{M}-x)\left(1-\frac{x}{\beta}\right)^{\frac{3}{2}}dx = 2\beta^3\left[\tfrac{1}{5}\left(\frac{\mathrm{M}}{\beta}-1\right)-\tfrac{1}{7}\left(\frac{\mathrm{M}}{\beta}-2\right)-\tfrac{1}{9}\right]$$

$$= \frac{2.2}{5.7}\beta^3\left[\frac{\mathrm{M}}{\beta}-\tfrac{4}{9}\right].$$

Therefore,

$$m = \tfrac{4}{9}\beta\,.\,\frac{\dfrac{\mathrm{M}}{\beta}-\tfrac{6}{11}}{\dfrac{\mathrm{M}}{\beta}-\tfrac{4}{9}},$$

or

$$\frac{m}{\mathrm{M}} = \tfrac{4}{9}\,.\,\frac{\dfrac{m'}{\mathrm{M}}}{1-\dfrac{\mathrm{v}^2}{\mathrm{v}_0^{\,2}}}\,.\,\frac{\dfrac{\mathrm{M}}{m'}\left(1-\dfrac{\mathrm{v}^2}{\mathrm{v}_0^{\,2}}\right)-\tfrac{6}{11}}{\dfrac{\mathrm{M}}{m'}\left(1-\dfrac{\mathrm{v}^2}{\mathrm{v}_0^{\,2}}\right)-\tfrac{4}{9}}. \quad \ldots\ldots \quad (66)$$

In order to compute from the formulæ, (59), (64), (66), it is necessary to make an assumption as to the value of $m'$ the mass of the meteorites of uniform size whose arrangement of density is supposed to be the same as that of the heterogeneous meteorites.

We have supposed that the law of frequency of masses is known at radius $a$, and that the mean mass is there equal to $\frac{1}{2}M$. Now, inside of that radius the larger masses are more frequent, and outside of it the smaller masses. I suppose, then, that throughout the isothermal sphere $m'$ lies half way between $m_0$ or $\frac{1}{2}$M and the maximum mass M, and in the adiabatic layer that it lies half way between $m_0$ or $\frac{1}{2}$M and the minimum mass 0.

Thus, inside I take $m' = \frac{3}{4}$M, and outside $m' = \frac{1}{4}M$.

As we only want to consider the general nature of the sorting process, these assumptions will suffice. It may also be remarked that a large variation of $m'$ is required to make any considerable difference in the numerical results.

We now have—

In the isothermal sphere (where $\mathrm{w}_0 = \frac{1}{3}\rho$),

$$\frac{\mathrm{M}}{m'} = \tfrac{4}{3}, \qquad \mathrm{P} = \log_e\left(\frac{\mathrm{w}}{\frac{1}{3}\rho}\right)^{\frac{4}{3}}, \qquad \tfrac{1}{2}\mathrm{M} = m_0;$$

In the adiabatic layer,

$$\frac{\mathrm{M}}{m'} = 4, \qquad Q = 1-\frac{\mathrm{M}}{m'}\left(1-\frac{\mathrm{v}^2}{\mathrm{v}_0^{\,2}}\right) = 4\frac{\mathrm{v}^2}{\mathrm{v}_0^{\,2}}-3.$$

Thus, our formulæ are :—

In the isothermal sphere, from (59),

$$\frac{m}{m_0} = \frac{2}{P}\cdot\frac{\left(\dfrac{\mathrm{w}}{\frac{1}{3}\rho}\right)^{\frac{4}{3}}(P^2-4P+6)-2(P+3)}{\left(\dfrac{\mathrm{w}}{\frac{1}{3}\rho}\right)^{\frac{4}{3}}(P-2)+(P+2)}; \quad \ldots\ldots \quad (67)$$

in the adiabatic layer,

when $\frac{v^2}{v_0^2} > \frac{3}{4}$, from (64),

$$\frac{m}{m_0} = \frac{\frac{4}{11} - \frac{4}{5}Q + \frac{9}{5}Q^{\frac{7}{2}} - 2Q^{\frac{9}{2}} + \frac{7}{11}Q^{\frac{11}{2}}}{\frac{1}{2}(1-Q)(1 - \frac{9}{5}Q + \frac{9}{5}Q^{\frac{7}{2}} - Q^{\frac{9}{2}})}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$

when $\frac{v^2}{v_0^2} < \frac{3}{4}$, from (66),

$$\frac{m}{m_0} = \frac{2}{9(1 - v^2/v_0^2)} \cdot \frac{\frac{19}{22} - v^2/v_0^2}{\frac{8}{9} - v^2/v_0^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

The values of $w/\frac{1}{3}\rho$ and of $v^2/v_0^2$ are tabulated in Table III., and from these I compute—

| | isothermal. | | | | | $\frac{v^2}{v_0^2} > \frac{3}{4}$ | | $\frac{v^2}{v_0^2} < \frac{3}{4}$ | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\frac{r}{a} =$ | 0 | ·16 | ·48 | ·80 | 1·0 | 1·09 | 1·2 | 1·33 | 1·5 | 1·71 | 2·00 | 2·21 | 2·46 | 2·79 |
| $\frac{m}{m_0} =$ | 1·41 | 1·38 | 1·22 | 1·11 | 1·0 | ·92 | ·83 | ·66 | ·49 | ·38 | ·30 | ·27 | ·24 | ·22 |

These values (together with two others in the isothermal part) are set out in fig. 2, and show the law of diminution of mean mass from centre to outside.

Fig. 2.

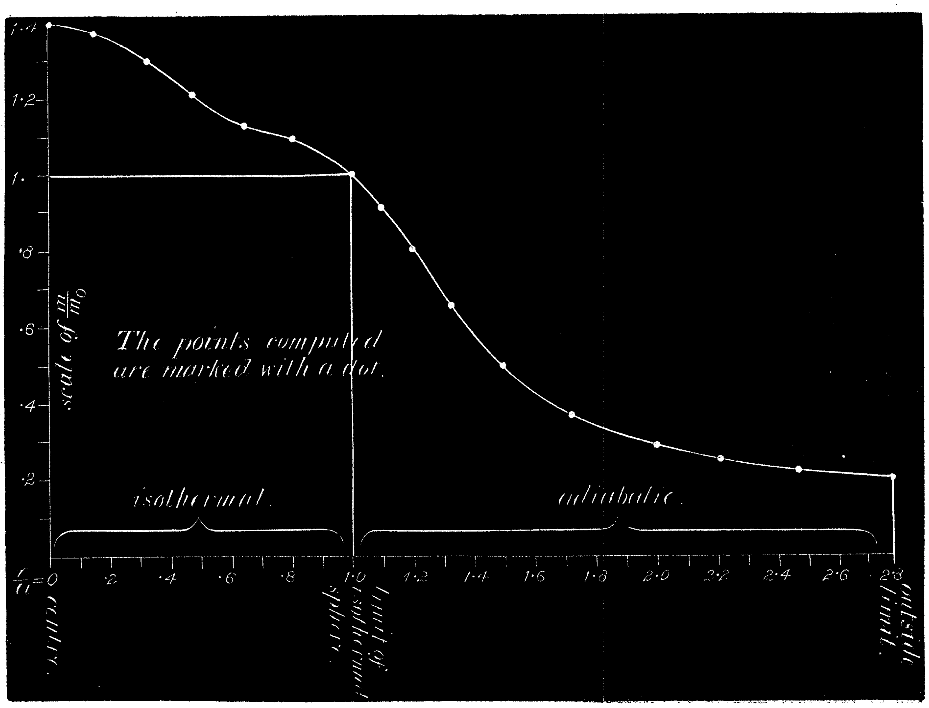



Diagram showing diminution of mean mass from centre to outside.

The evaluation of mean mass in the fringe (see § 13), where collisions are supposed to be non-existent, is not very difficult, although it involves some troublesome algebra. I do not give the investigation, merely remarking that it leads to almost exactly the same kind of law of diminution of mean mass as we have found in the adiabatic layer.

## § 17. *Summary.*

The first and second sections only involved arguments of a general character in which mathematical analysis was unnecessary. The reader who does not wish to concern himself with details may therefore be supposed to have passed from §§ 1 and 2 to this Summary.

In order to submit the theory to an adequate test, it is necessary to discuss some definite case of the aggregation of a swarm of meteorites, and it is obvious that the only system of which we possess any knowledge is our own. It is accordingly supposed that a number of meteorites have fallen together from a condition of wide dispersion, and have ultimately coalesced so as to leave the Sun and planets as their progeny. The object of this paper is to consider the mechanical condition of the system after the cessation of any considerable supply of meteorites from outside, and before the coalescence of the swarm into a star with attendant planets.

For the sake of simplicity, the meteorites are considered to be spherical, and are treated, at least in the first instance, as being of uniform size.

It is assumed provisionally that the kinetic theory of gases may be applied for the determination of the distribution of the meteorites in space. No account being taken of the rotation of the system, the meteorites will be arranged in concentric spherical layers of equal density of distribution, and the quasi-gas, whose molecules are meteorites, being compressible, the density will be greater towards the centre of the swarm.

The elasticity of a gas depends on the kinetic energy of agitation of its molecules; and, therefore, in order to determine the law of density in the swarm, we must know the distribution of kinetic energy of agitation. It is assumed that, when the swarm comes under our notice, uniformity of distribution of energy has been attained throughout a central sphere, which is surrounded by a layer of meteorites with that distribution of kinetic energy which in a gas corresponds to convective equilibrium. In other words, we have a quasi-isothermal sphere surrounded by what may be called an atmosphere in convective equilibrium, and with continuity of density and velocity of agitation at the sphere of separation. Since in a gas in convective equilibrium the law connecting pressure and density is that which holds when the gas is contained in a vessel impermeable to heat, such an arrangement of gas has been called by M. RITTER "an isothermal-adiabatic sphere," and the same term is adopted here as applicable to a swarm of meteorites. The justifiability of these assumptions will be considered later.

The first problem which presents itself, then, is the equilibrium of an isothermal sphere of gas under its own gravitation. The law of density is determined in § 4; but it will here suffice to remark that, if a given mass be enclosed in an envelope of given radius, there is a minimum temperature (or energy of agitation) at which isothermal equilibrium is possible. The minimum energy of agitation is found to be such that the mean square of velocity of the meteorites is almost exactly $\frac{6}{5}$ (viz. 1·1917) of the square of the velocity of a satellite grazing the surface of the sphere in a circular orbit.

As indicated above, it is supposed that in the meteor-swarm the rigid envelope bounding the isothermal sphere is replaced by a layer of meteorites in convective equilibrium. The law of density in the adiabatic layer is determined in § 5, and it appears that, when the isothermal sphere has minimum temperature, the mass of the adiabatic atmosphere is a minimum relatively to that of the isothermal sphere. Numerical calculation shows, in fact, that the isothermal sphere cannot amount in mass to more than 46 per cent. of the mass of the whole isothermal-adiabatic sphere, and that the limit of the adiabatic atmosphere is at a distance equal to 2·786 times the radius of the isothermal sphere.* A table of various quantities in such a system, at various distances from the centre, is given in Table III., § 6.

It is next proved, in § 7, that the total energy, existing in the form of energy of agitation in an isothermal-adiabatic sphere, is exactly one-half of the potential energy lost in the concentration of the matter from a condition of infinite dispersion. This result is brought about by a continual transfer of energy from a molar to a molecular form, for a portion of the kinetic energy of a meteorite is constantly being transferred into the form of thermal energy in the volatilised gases generated on collision. The thermal energy is then lost by radiation.

It is impossible as yet to sum up all the considerations which go to justify the assumption of the isothermal-adiabatic arrangement; but it is clear that uniformity of kinetic energy of agitation in the isothermal sphere must be principally brought about by a process of diffusion. It is, therefore, interesting to consider what amount of inequality in the kinetic energy would have to be smoothed away.

The arrangement of density in the isothermal-adiabatic sphere being given, it is easy to compute what the kinetic energy would be at any part of the swarm, if each meteorite fell from infinity to the neighbourhood where we find it, and there retained all the velocity due to such fall. The variation of the square of this velocity gives an indication of the amount of inequality of kinetic energy which has to be degraded by conversion into heat and redistributed by diffusion in the attainment of uniformity. This may be called "the theoretical value of the kinetic energy"; it is tabulated in Table III., on the line called "square of velocity of satellite." It rises from zero at the centre of the sphere to a maximum, which is attained nearly half way through the adiabatic layer, and then falls again. If the radius of the isothermal sphere be unity, then from $\frac{1}{2}$ to 2 the variations of this theoretical value of the kinetic energy

* These results had been previously discovered by M. Ritter.

are small. Since this "theoretical value of the kinetic energy" is zero at the centre, there must have been diffusion of energy from without inwards, and considerations of the same kind show that when a planet consolidates there must be a cooling of the middle strata both outwards and inwards.

We must now consider the nature of the criterion which determines whether the hydrodynamical treatment of a swarm of meteorites is permissible.

The hydrodynamical treatment of an ideal plenum of gas leads to the same result as the kinetic theory with regard to any phenomenon involving purely a mass, when that mass is a large multiple of the mass of a molecule; to any phenomenon involving purely a length, when the cube of that length contains a large number of molecules; and to any phenomenon involving purely a time, when that time is a large multiple of the mean interval between collisions. Again, any velocity to be justly deduced from hydrodynamical principles must be expressible as the edge of a cube containing many molecules passed over in a time containing many collisions of a single molecule; and a similar statement must hold of any other function of mass, length, and time.

Beyond these limits, we must go back to the kinetic theory itself, and in using it care must be taken that enough molecules are considered at once to impart statistical constancy to their properties.

There are limits, then, to the hydrodynamical treatment of gases, and the like must hold of the parallel treatment of meteorites.

The principal question involved in the nebular hypothesis seems to be the stability of a rotating mass of gas; but, unfortunately, this has remained up to now an untouched field of mathematical research. We can only judge of probable results from the investigations which have been made concerning the stability of a rotating mass of liquid. Now, it appears that the instability of a rotating mass of liquid first enters through the graver modes of gravitational oscillation. In the case of a rotating spheroid of revolution the gravest mode of oscillation is an elliptic deformation, and its period does not differ much from that of a satellite which revolves round the spheroid so as to graze its surface. Hence, assuming for the moment that a kinetic theory of liquids had been formulated, we should not be justified in applying the hydrodynamical method to this discussion of stability unless the periodic time of such a satellite were a large multiple of the analogue of the mean free time of a molecule of liquid.*

Carrying, then, this conclusion on to the kinetic theory of meteorites, it seems probable that hydrodynamical treatment must be inapplicable for the discussion of such a theory as the meteoric-nebular hypothesis, unless a similar relation holds good.

These considerations, although of a very general character, will afford a criterion of the applicability of hydrodynamics to the discussion of the mechanical conditions of a swarm of meteorites in the kind of problem suggested by the nebular hypothesis.

* If the molecules of liquid describe orbits about one another, the analogue would probably be the mean periodic time of one molecule about another.

In § 9 two criteria, suggested by this line of thought, are found. They measure, roughly speaking, the degree of curvature of the average path pursued by a meteorite between two collisions. These two criteria, denoted $D/L$ and $A/C$, will afford a measure of the applicability of hydrodynamics in the sense above indicated.

After these preliminary investigations, we have to consider what kind of meeting of two meteorites will amount to an "encounter" within the meaning of the kinetic theory. Is it possible, in fact, that two meteorites can considerably bend their paths under the influence of gravitation when they pass near one another? This question is answered in § 8, where a formula is found for the deflection of the path of each of a pair of meteorites, when, moving with their mean relative velocity, they graze past one another without striking. It appears from the formula that, unless they have the dimensions of small planets, the mutual gravitational influence is practically insensible. Hence, nothing short of absolute impact is to be considered an encounter in the kinetic theory; and what is called the radius of "the sphere of action" is simply the distance between the centres of a pair when they graze, and is, therefore, the sum of their radii, or, if of uniform size, the diameter of one of them.

The next point to consider is the mass and size which must be attributed to the meteorites.

The few samples which have been found on the earth prove that no great error can be committed if the average density of a meteorite be taken as a little less than that of iron, and I accordingly suppose their density to be six times that of water.

Undoubtedly, in a swarm of meteorites all sizes exist (a supposition considered hereafter); for, even if originally of one uniform size, they would, by subsequent fracture, be rendered diverse. But in the first consideration of the problem they have been treated as of uniform size, and, as actual average sizes are nearly unknown, results are given in the numerical table for meteorites weighing $3\frac{1}{8}$ grammes. By merely shifting the decimal point one, two, or three places to the right the results become applicable to meteorites weighing $3\frac{1}{8}$ kilogrammes, $3\frac{1}{8}$ tonnes, 3125 tonnes, and so on.

It is known that meteorites are actually of irregular shapes, but certainly no material error can be incurred when we treat them as being spheres.

The object of all these investigations is to apply the formulæ to a concrete example. The mass of the system is therefore taken as equal to that of the Sun, and the limit of the swarm at any arbitrary distance from the present Sun's centre. The theory is, of course, most severely tested the wider the dispersion of the swarm; and, accordingly, in the numerical example the outside limit of the Solar swarm is taken at $44\frac{1}{2}$ times the Earth's distance from the Sun, or further beyond the planet Neptune than Saturn is from the Sun. This assumption makes the limit of the isothermal sphere at distance 16, about half way between Saturn and Uranus.

The results, applicable to meteorites of $3\frac{1}{8}$ grammes, are exhibited in Table IV., § 10.

The velocity of mean square in the isothermal sphere is $\sqrt{(6/5)}$ of the linear velocity of a planet at distance 16, revolving about a central body with a mass equal to 46 per cent. of that of the Sun, viz., $5\frac{1}{3}$ kilometres per second; and in the adiabatic layer it diminishes down to zero at distance $44\frac{1}{2}$. This velocity is independent of the size of the meteorites.

The mean free path between collisions ranges from 42,000 kilometres at the centre, to 1,300,000 kilometres at radius 16, and to infinity at radius $44\frac{1}{2}$. The mean interval between collisions ranges from a tenth of a day at the centre, to three days at radius 16, and to infinity at radius $44\frac{1}{2}$. The criterion $D/L$ ranges from $\frac{1}{60,000}$ at the distance of the asteroids, to $\frac{1}{3600}$ at radius 16, and to infinity at radius $44\frac{1}{2}$. The criterion $A/C$ is somewhat smaller than $D/L$. All these quantities are ten times as great for meteorites of $3\frac{1}{8}$ kilogrammes, and a hundred times as great for meteorites of $3\frac{1}{8}$ tonnes.

From a consideration of the table it appears that, with meteorites of $3\frac{1}{8}$ kilogrammes, the collisions are sufficiently frequent, even beyond the orbit of Neptune, to allow the kinetic theory to be applicable in the sense explained. But, if the meteorites weigh $3\frac{1}{8}$ tonnes, the criteria cease to be very small about distance 24; and, if they weigh 3125 tonnes, they cease to be very small at about the orbit of Jupiter.

It may be concluded, then, that, as far as frequency of collision is concerned, the hydrodynamical treatment of a swarm of meteorites is justifiable.

Although these numerical results are necessarily affected by the conjectural values of the mass and density of the meteorites, yet it was impossible to arrive at any conclusion whatever as to the validity of the theory without numerical values, and such a discussion as the above was therefore necessary. If the particular values used are not such as to commend themselves to the judgment of the reader, it is easy to substitute others in the formulæ, and so submit the theory to another test.

I now pass on to consider some results of this view of a swarm of meteorites, and to consider the justifiability of the assumption of an isothermal-adiabatic arrangement of density.

With regard to the uniformity of distribution of kinetic energy in the isothermal sphere, it is important to ask whether or not sufficient time can have elapsed in the history of the system to allow of the equalisation by diffusion.

In § 11 the rate of diffusion of the kinetic energy of agitation is considered, and it is shown that, in the case of our numerical example, primitive inequalities of distribution would, in a few thousand years, be sensibly equalised over a distance some ten times as great as our distance from the Sun. This result, then, goes to show that we are justified in assuming an isothermal sphere as the centre of the swarm. As, however, the swarm contracts, the rate of diffusion diminishes as the inverse $\frac{5}{2}$ power of its linear dimensions, whilst the rate of generation of inequalities of distribution of kinetic energy, through the imperfect elasticity of the meteorites, increases. Hence, in a late stage of the swarm inequalities of kinetic energy would be set up; thus, there would be a tendency to the production of convective currents, and the whole swarm would probably settle down to the condition of convective equilibrium throughout.

It may be conjectured, then, that the best hypothesis in the early stages of the swarm is the isothermal adiabatic arrangement, and later an adiabatic sphere. It has not seemed to me worth while to discuss the latter hypothesis in detail at present.

The investigation of § 11 also gives the coefficient of viscosity of the quasi-gas, and shows that it is so great that the meteor-swarm must, if rotating, revolve nearly without relative motion of its parts, other than the motion of agitation. But, as the viscosity diminishes when the swarm contracts, this would probably not be true in the later stages of its history, and the central portion would probably rotate more rapidly than the outside. It forms, however, no part of the scope of this paper to consider the rotation of the system.

In § 12 the rate of loss of kinetic energy through imperfect elasticity is considered, and it appears that the rate estimated per unit time and volume must vary directly as the square of the quasi-pressure and inversely as the mean velocity of agitation. Since the kinetic energy lost is taken up in volatilising solid matter, it follows that the heat generated must follow the same law. The mean temperature of the gases generated in any part of the swarm depends on a great variety of circumstances, but it seems probable that its variation would be according to some law of the same kind. Thus, if the spectroscope enables us to form an idea of the temperature in various parts of a nebula, we shall at the same time obtain some idea of the distribution of density.

It has been assumed that the outer portion of the swarm is in convective equilibrium, and therefore there is a definite limit beyond which it cannot extend. Now, a medium can only be said to be in convective equilibrium when it obeys the laws of gases, and the applicability of those laws depends on the frequency of collisions. But at the boundary of the adiabatic layer the velocity of agitation vanishes, and collisions become infinitely rare. These two propositions are mutually destructive of one another, and it is impossible to push the conception of convective equilibrium to its logical conclusion. There must, in fact, be some degree of rarity of density, and of collisions, at which the statistical treatment of the medium breaks down.

I have sought to obtain some representation of the state of things by supposing that collisions never occur beyond a certain distance from the centre of the swarm. Then, from every point of the surface of the sphere, which limits the regions of collisions, a fountain of meteorites is shot out, in all azimuths and inclinations to the vertical, and with velocities grouped about a mean according to the law of error. These meteorites ascend to various heights without collision, and, in falling back on to the limiting sphere, cannonade its surface, so as to counterbalance the hydrostatic pressure at the limiting sphere.

The distribution of meteorites, thus shot out, is investigated in § 13, and it is found that near the limiting sphere the decrease in density is somewhat more rapid than the decrease corresponding to convective equilibrium. But at more remote distances the decrease is less rapid, and the density ultimately tends to vary inversely as the square of the distance from the centre.

It is clear, then, that, according to this hypothesis, the mass of the system is infinite in a mathematical sense, for the existence of meteorites with nearly parabolic and hyperbolic orbits necessitates an infinite number, if the loss of the system shall be made good by the supply.*

But, if we consider the subject from a physical point of view, this conclusion appears unobjectionable. The ejection of molecules with exceptionally high velocities from the surface of a liquid is called evaporation, and the absorption of others is called condensation. The general history of a swarm, as stated in § 2, may then be put in different words, for we may say that at first a swarm gains by condensation, that condensation and evaporation balance, and, finally, that evaporation gains the day.

If the hypothesis of convective equilibrium be pushed to its logical conclusion, we reach a definite limit to the swarm, whereas, if collisions be entirely annulled, the density goes on decreasing inversely as the square of the distance. The truth must clearly lie between these two hypotheses. It is thus certain that even the very small amount of evaporation shown by the formulæ derived from the hypothesis of no collisions must be in excess of the truth; and it may be that there are enough waifs and strays in space, ejected from other systems, to make up for the loss. Whether or not the compensation is perfect, a swarm of meteorites would pursue its evolution without being sensibly affected by a slow evaporation.

Up to this point the meteorites have been considered as of uniform size, but it is well to examine the more truthful hypothesis, that they are of all sizes, grouped about a mean according to a law of error.

It appears, from the investigation in § 14, that the larger stones move slower, the smaller ones faster; and the law is that the mean kinetic energy is the same for all sizes.

It is proved that the mean path between collisions is shorter in the proportion of 7 to 11, and the mean frequency greater in the proportion of 4 to 3, than if the meteorites were of uniform mass, equal to their mean. Hence, the previous numerical results for uniform size are applicable to non-uniform meteorites of mean mass about a third greater than the uniform mass; for example, the results for uniform meteorites of $3\frac{1}{8}$ tonnes apply to non-uniform ones of mean mass, a little over 4 tonnes.

The means here spoken of refer to all sizes grouped together, but there are a separate mean free path and a mean frequency appropriate to each size. These are investigated

[* It must also be borne in mind that the very high velocities, which occur occasionally in a medium with perfectly elastic molecules, must happen with great rarity amongst meteorites. An impact of such violence that it *ought* to generate a hyperbolic velocity will probably merely cause fracture.—Added Nov. 23, 1888.]

in § 15, and their various values are illustrated in fig. 1. The horizontal scale in that figure gives the ratio of the radius of each size to the radius of the meteorite of mean mass. The vertical scales are the ratio of the mean free path of any size to that of all sizes together, and the ratio of the mean frequency for any size to that of all sizes together. The figure shows that collisions become infinitely frequent for the infinitely small ones, because of their infinite velocity; and again infinitely frequent for the infinitely large ones, because of their infinite size. There is a minimum frequency of collision for a certain size, a little less in radius than the mean, and considerably less in mass than the mean mass.

For infinitely small meteorites, the mean free path reaches a finite limit, equal to about four times the grand mean free path; but this could not be shown in the figure without a considerable extension of it upwards. For infinitely large ones, the mean free path becomes infinitely short. It must be borne in mind that there are infinitely few of the infinitely large and small meteorites.

Variety of size does not, then, so far, materially affect the results.

But a difference arises when we come to consider the different parts of the swarm. The larger meteorites, moving with smaller velocities, form a quasi-gas of less elasticity than do the smaller ones. Hence, the larger meteorites are more condensed towards the centre than are the smaller ones, or the large ones have a tendency to sink down, whilst the small ones have a tendency to rise. Accordingly, the various kinds are to some extent sorted according to size.

In § 16, an investigation is made of the mean mass of the meteorites at various distances from the centre, both inside and outside of the isothermal sphere, and fig. 2 is drawn to illustrate the law of diminution of mean mass.

It is also clear that the loss of the system through evaporation must fall more heavily on the small meteorites than on the large ones.

After the foregoing summary, it will be well to briefly recapitulate the principal conclusions which seem to be legitimately deducible from the whole investigation; and, in this recapitulation, qualifications must necessarily be omitted, or stated with great brevity.

When two meteorites are in collision, they are virtually highly elastic, although ordinary elasticity must be nearly inoperative.

A swarm of meteorites is analogous with a gas, and the laws governing gases may be applied to the discussion of its mechanical properties. This is true of the swarm from which the Solar system was formed, when it extended beyond the orbit of the planet Neptune.

When the swarm was very widely dispersed, the arrangement of density and of velocity of agitation of the meteorites was that of an isothermal-adiabatic sphere. Later in its history, when the swarm had contracted, it was probably throughout in convective equilibrium.

The actual mean velocity of the meteorites is determinable in a swarm of given mass, when expanded to a given extent.

The total energy of agitation in an isothermal-adiabatic sphere is half the potential energy lost in the concentration from a condition of infinite dispersion.

The half of the potential energy lost, which does not reappear as kinetic energy of agitation, is expended in volatilising solid matter and heating the gases produced on the impact of meteorites. The heat so generated is gradually lost by radiation.

The amount of heat generated per unit time and volume varies as the square of the quasi-hydrostatic pressure, and inversely as the mean velocity of agitation. The temperature of the gases volatilised probably varies by some law of the same nature.

The path of the meteorites is approximately straight, except when abruptly deflected by a collision with another. This ceases to be true at the outskirts of the swarm, where the collisions have become rare. The meteorites here describe orbits, under gravity, which are approximately elliptic, parabolic, and hyperbolic.

In this fringe to the swarm the distribution of density ceases to be that of a gas under gravity, and, as we recede from the centre, the density at first decreases more rapidly, and afterwards less rapidly, than if the medium were a gas.

Throughout all stages of the history of a swarm there is a sort of evaporation, by which the swarm very slowly loses in mass, but this loss is more or less counter-balanced by condensation. In the early stages, the gain by condensation outbalances the loss by evaporation; they then equilibrate; and, finally, the evaporation may be greater than the condensation.

Throughout the swarm the meteorites are partially sorted, according to size. As we recede from the centre, the number of small ones preponderates more and more and, thus, the mean mass continually diminishes with increasing distance. The loss to the system by evaporation falls principally on the smaller meteorites.

A meteor-swarm is subject to gaseous viscosity, which is greater the more widely diffused is the swarm. In consequence of this, a widely extended swarm, if in rotation, will revolve like a rigid body, without relative movement of its parts. Later in its history, the viscosity will, probably, not suffice to secure uniformity of rotation, and the central portion will revolve more rapidly than the outside.

[The kinetic theory of meteorites may be held to present a fair approximation to the truth in the earlier stages of the evolution of the system. But ultimately the majority of the meteors must have been absorbed by the central Sun and its attendant planets, and amongst the meteors which remain free the relative motion of agitation must have been largely diminished. These free meteorites—the dust and refuse of the system—probably move in clouds, but with so little remaining motion of agitation that (except, perhaps, near the perihelion of very eccentric orbits) it would scarcely be permissible to treat the cloud as in any respects possessing the mechanical properties of a gas.*]

The value of this whole investigation will appear very different to different minds. To some it will stand condemned, as altogether too speculative; others may think that

* Added Nov. 23, 1888.

it is better to risk error on the chance of winning truth. To me, at least, it appears that the line of thought flows in a true channel; that it may help to give a meaning to the observations of the spectroscopist; and that many interesting problems, here barely alluded to, may, perhaps, be solved with sufficient completeness to throw light on the evolution of nebulæ and of planetary systems.



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